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Rapports de Recherche

N° 264

ON THE GLOBAL SET OF SOLUTIONS OF A NONLINEAR O.D.E.: THEORETICAL AND NUMERICAL DESCRIPTION

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Janvier 1984

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OF A NONLINEAR ODE :

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by

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RESUME : On étudie dans cette note le problème aux limites non linéaire de la forme : $-u'' = f(u) + \lambda g(x)$ dans $[0,1]$ avec $u(0) = u(1) = 0$ où f est une fonction convexe croissante, avec $f(0) = 0$, $f'(0) = 0$, $\lim_{s\to\pm\infty} \frac{f(s)}{s} = \infty$ et g est une fonction positive dans $[0,1]$. Dans le cas particulier, où $f(u) = |u|^{p-1}u$ et $g(x) \equiv 1$, on obtient une description complète et globale des solutions de ce problème (infinité des points de bifurcation et de retournement). Certains de ces résultats, concernant les solutions positives peuvent être étendus dans le cas d'équations du type elliptique semi-linéaire en dimension supérieure.

ABSTRACT : We study non-linear O.D.E.'s of the type $-u'' = f(u) + \lambda g(x)$, with Dirichlet boundary conditions. Here f is a convex non-decreasing function with $f(0) = 0$, $f'(0) = 0$, $\lim_{s\to\infty} \frac{f(s)}{s} = \infty$ and g is a positive function in $(0,1)$. In the special case $f(u) = |u|^{p-1} u$, and $g(x) \equiv 1$, a complete and global description the set of solutions is given. (infinite number of bifurcation points and turning points). Some of the results regarding the positive solutions are shown to extend for a similar semi-linear elliptic problem in higher dimensions.

This work was carried out when the author was in INRIA under deputation from T.I.F.R., India.

PART - I

THEORETICAL STUDY

I - INTRODUCTION

In this article we study the behaviour of the set of solutions to the problem :

$$(1.1) \qquad \begin{cases} -u'' = f(u) + \lambda g(x) \text{ on } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Here f is a convex, non-decreasing function with $f(0) = 0$, $f'(0) = 0$, super linear at ∞ and g a non negative function in $[0,1]$, strictly positive in $(0,1)$. For $f(u) = |u|^{p-1}u$ for $p > 1$, and $g(x) \equiv 1$, the global behaviour of the solutions is established rigorously and also obtained by numerical computation via a continuation method. Some of the results concerning the branch of positive solutions extend to the semi-linear problem :

$$(1.2) \qquad \begin{cases} -\Delta u = f(u) + \lambda g(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, is a smooth bounded domain.

For $N \geq 1$, $\lambda = 0$ and $f(u) = |u|^{p-1}u$, it is known (cf [16]) that there does not exist non trivial solution to (1.2), when Ω is star-shaped and $p \geq \frac{N+2}{N-2}$. For $p < \frac{N+2}{N-2}$, existence of infinitely many solutions can be proved using Lusternik-Schnirelmann theory (see for example [17]).

In the case $\lambda \neq 0$, in one-dimension for a function f with,

$$\lim_{s \to \pm\infty} \frac{f(s)}{s} = +\infty,$$

EHRMANN [8] and FUČIK & LOVICAR [9] show the existence of infinitely many solutions. In fact, the results of [8] and [9] are more general, allowing for a right hand side of the type of $f(x,u)$. For $\lambda \neq 0$, $N > 1$, for $f(u) = |u|^{p-1}u$, BAHRI & BERESTYCKI [2] get infinitely many solutions provided $p \leq p_N$, for a certain $1 < p_N < \frac{N+2}{N-2}$. In the case $f(u) = u^3$, a necessary and sufficient condition for the existence of positive solution, is obtained by BARAS & PIERRE [3]. The equation :

$$(1.3) \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been studied by a number of authors. Depending on the type of non linearity f, the branch of positive solutions is unbounded or turns back etc. A complete and up-to-date presentation of these results can be found in the survey by P.L. LIONS [14]. We would like to refer also to the work of PEITGEN & SCHMITT [15], where a two parameter problem is studied :

$$(1.4) \qquad \begin{cases} \mathcal{L}u + \lambda f(u) = 0 & \text{in } \Omega \subset \mathbb{R}^N, \\ u = \sigma\phi & \text{on } \partial\Omega, \end{cases}$$

where f is asymptotically linear and $\mathcal{L}$ an elliptic operator. Since the parameter σ here plays a somewhat similar role as the parameter λ in (1.1), the bifurcation diagram for the O.D.E. model of (1.4) exhibits a bifurcation structure, very much related to the one, observed in our case.

In one dimension, more precise information is obtained for the equation (1.1) when $\lambda = 0$, in [4] and [7]. There exists exactly one solution $u_k^{\pm} \in \Sigma_k^{\pm}$ where :

$$\Sigma_k = \{v \in C_0^1(0,1) \; ; \; v \text{ has } (k-1) \text{ nodes in } (0,1)\}^*,$$

$$\Sigma_k^+ = \{v \in \Sigma_k \; ; \; v'(0) > 0\} \; ; \; \Sigma_k^- = -\Sigma_k^+.$$

* node is a simple zero. Note that for $\lambda = 0$, a non-trivial solution can have only simple zeroes.

Now the natural question is how this set of solutions changes when the term $\lambda g(x)$ is added to $f(u)$. That is to see whether there is a branch passing through each solution for $\lambda = 0$, and what is its global behaviour.

The motivation to study such problems, comes from the fact that this type of equations appears naturally in many physical phenomena. It also arises in the work of J.L. LIONS (cf. [13]) on optimal control of systems with multiple states.

Our main result concerns the equation :

$$(1.5)\qquad \begin{cases} -u'' = |u|^{p-1}u + \lambda \text{ in } (0,1), \\ u(0) = 0 = u(1). \end{cases}$$

We use the following notations :

$$C_0^1(0,1) = \{u \in C^1(0,1)\ ;\ u(0) = u(1) = 0\},$$

$$X = \{(\lambda,u) \in \mathbb{R} \times C_0^1(0,1)\ ;\ (\lambda,u) \text{ satisfies } (1.5)\},$$

$$E = \{(\lambda,u) \in X\ ;\ u'(0) = u'(1) = 0\},$$

$$S_k^+ = \text{clo}\{(\lambda,u) \in X\ ;\ u \text{ has exactly } (k-1) \text{ nodes in } (0,1)\ ;\ u'(0) > 0\},$$

$$S_k^- = -S_k^+,$$

$$S_k = S_k^+ \cup S_k^-.$$

Now our main result is :

<u>THEOREM 1.1</u> : For the equation (1.5), $\forall\, p > 1$, we have,

(i) S_k is bounded in X for all $k \in \mathbb{N}$.

(ii) S_1 is connected ; each S_{2k+1}, for $k \geq 1$ consists exactly of two connected components S_{2k+1}^+ and S_{2k+1}^- which are C^1 curves.

(iii) Each S_{2k} is a closed C^1 curve.

(iv) There exist infinitely many bifurcation points, $\{P_{2k}, Q_{2k}\}$ for $k \in \mathbb{N}$, where $Q_{2k} = -P_{2k}$.

Furthermore,

(a) $P_{2k} = (\lambda_{2k}, u_{2k})$ with $u_{2k} \in E$,

(b) $E \cap S^{+}_{2k+1} = \{P_{2k}, Q_{2k+2}\}$,

(c) $E \cap S^{-}_{2k+1} = \{Q_{2k}, P_{2k+2}\}$,

(d) $E \cap S^{\pm}_{k} = \{P_{2k}, Q_{2k}\}$, for $k \geq 1$.

(v) Each $S^{\pm}_{k}$ possesses at least one turning point $R^{\pm}_{k}$ for $k \geq 1$.

REMARK 1.2 : In fact an explicit expression for P_{2k}, Q_{2k} can be obtained by an appropriate integration and a change of variable :

$$P_{2k} = (\lambda_{2k}, u_{2k}) \; ; \; Q_{2k} = -P_{2k},$$
$$\lambda_{2k} = (2kI_p)^p \; ; \; I_p = \int_0^{(p+1)^{1/p}} \frac{dt}{\sqrt{2t - \frac{2t^{p+1}}{p+1}}} .$$

REMARK 1.3 : The results in the theorem are illustrated by the global picture, obtained numerically, for the case $f(u) = u^3$. The bifurcation diagram consists of 2 spirals starting from 0, corresponding to the branches S_{2k+1}, $k \geq 0$, cut by a series of closed curves, corresponding to the branches S_{2k}, $k \geq 1$.

REMARK 1.4 : For the case $f(u) = u^3$, the results annouced in the theorem can be obtained directly, by a phase plane argument, somewhat similar to the one, used by TESAI & DE MOTTONI (cf.[7]). The author thanks Professor A.A. MINZONI, for illuminating discussions in this direction.

In the following sections, we prove Theorem (1.1) and obtain the extension of a few results concerning the branch of positive solutions to the equation (1.2) in higher dimensions. Also refer to [5] and [14] for similar results and to [12] for results in one dimension.

In part-II, a description of the numerical scheme used to follow up the branches, is given. The numerical results are also presented.

The author is grateful to Professor H. BERESTYCKI, for suggesting the problem and for many fruitful discussions.

2 - A MODEL PROBLEM

To get an insight into the result, we first analyse a simpler equation :

$$(2.1) \qquad \begin{cases} -\Delta u = (\int u^2)^\alpha . u + h(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \ \alpha > 0. \end{cases}$$

The global behaviour of the solutions of this equation for certain functions h, is indeed a prototype of the behaviour described in Theorem (1.1).

For $h(x) = 0$, (2.1) is an eigenvalue problem and the only solutions are :

$$u = \pm(\mu_k)^{1/2\alpha}\phi_k,$$

where ϕ_k satisfies,

$$\begin{cases} -\Delta\phi_k = \mu_k\phi_k & \text{in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega, \\ \int_\Omega \phi_k^2 = 1. \end{cases}$$

Let us consider the case $h = \lambda\phi_j$. First we look for solutions of the type $u = t_j\phi_j$. Substituting in the equation, we obtain,

$$\lambda = \mu_j t_j - t_j^{2\alpha+1}.$$

Thus depending on the value of λ, we have one, two or three solutions. The branch S_j in λ-u plane could be represented as follows :

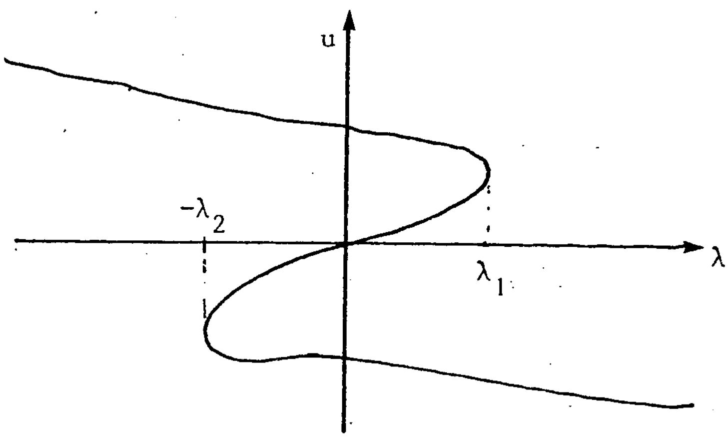


FIGURE 1

If μ_j is an eigenvalue of multiplicity $p > 1$, and ϕ_j is the normalised eigenvector in the corresponding eigenspace of dimension p, then again u is uniquely determined just as in the previous case and we get a similar solution curve S_j.

Next we look for other solutions of the type,

$$u = t_j\phi_j + t_k\phi_k.$$

A direct substitution into the equation and projection onto the eigenspaces lead to the equations :

$$\begin{cases} \mu_j t_j = (t_j^2 + t_k^2)^\alpha t_j + \lambda, \\ \mu_k t_k = (t_j^2 + t_k^2)^\alpha t_k. \end{cases}$$

For $t_k \neq 0$,

$$t_j = \{\frac{\lambda}{\mu_j - \mu_k}\} \; ; \; t_k = (\mu_k^{1/\alpha} - (\frac{\lambda}{\mu_j - \mu_k})^2)^{1/2}.$$

This shows that (t_j, t_k) forms an ellipse S_k, which cuts S_j for the value $t_k = 0$. Even if μ_j is an eigenvalue of multiplicity $p > 1$, the same behaviour persists.

It can be checked easily that there can be at most only one component ϕ_k different from ϕ_j. Then, the only solutions, corresponding to :

$$h = \lambda\phi_j,$$

are :

$$\begin{cases} u = t_j\phi_j, \\ u = t_j\phi_j + t_k\phi_k. \end{cases}$$

The bifurcation diagram consists of an unbounded curve S_j, cut by ellipses S_k, which is somewhat similar to the behaviour described in the Theorem. We can also analyse the case ,

$$h = \sum_{j=1}^{m} \lambda_j\phi_j,$$

which again leads to a bifurcation diagram similar to the above one.

3 - POSITIVE SOLUTIONS IN DIMENSION $N \geq 1$

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and q be a continuous function on Ω. Let $\mu_1 = \mu_1(q,\Omega)$ denote the first eigen value of the problem :

$$\begin{cases} -\Delta u + q(x)u = \mu_1 u \text{ in } \Omega, \\ \qquad\qquad u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Thus $\mu_1(0,\Omega)$ denotes the first eigenvalue of the Laplacian. Let $f = f(s)$ be a C^1 function, $f : \mathbb{R} \to \mathbb{R}$, verifying,

(3.1) $\qquad f(0) = 0 \ ; \ f'(0) = 0,$

(3.2) $\qquad$ f is convex, non decreasing,

(3.3) $\qquad \lim\limits_{s \to \pm\infty} \dfrac{f(s)}{s} = +\infty.$

Let g be a continuous function $g : \Omega \to \mathbb{R}$ such that :

(3.4) $\qquad g > 0$ on Ω.

The following classical Lemma will be crucial for our proof.

LEMMA 3.1 : There does not exist a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, satisfying :

$$(3.5) \qquad \begin{cases} -\Delta u + qu > \mu_1 u \text{ in } \Omega, \text{ for } q \in C^0(\Omega) \quad , \\ \qquad\qquad u \geq 0 \quad \text{on } \partial\Omega \ , \end{cases}$$

where $\mu_1 = \mu_1(q,\Omega)$.

PROOF : We recall that the first eigenfunction ϕ of the following problem can be chosen to be positive in Ω :

$$(3.6) \qquad \begin{cases} -\Delta\phi + q\phi = \mu_1\phi \text{ in } \Omega, \\ \qquad\qquad q = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Supposing that there exists a function u satisfying (3.5), we get by multiplying (3.5) by ϕ and integrating by parts and using (3.6),

$$(3.7) \qquad \int_{\partial\Omega} u \frac{\partial\phi}{\partial n} > 0.$$

Since $\frac{\partial\phi}{\partial n} < 0$ on $\partial\Omega$, we get a contradiction. ∎

Next we show that there are no positive solutions for large values of λ. Analogous results are known for the equation (1.3) (See for example [14]).

PROPOSITION 3.2 : Under the assumptions (3.1) to (3.4), there exists a constant M, $0 < M < \infty$, such that there exists no positive solution for the equation (1.2) for $\lambda > M$.

PROOF : To obtain the proof by contradiction, we suppose that there exists (λ_j, u_j), a sequence of positive solutions for (1.2) with $\lambda_j \to \infty$. Fix a proper subset $\Omega_0 \subset \Omega$ and let $\mu_1 = \mu_1(0, \Omega_0)$.

We claim that u_j goes to ∞ uniformly over compact subsets of Ω. In fact for any $\Omega_1 \subset K \subset \Omega$, Ω_1 open and K compact, $g > 0$ in Ω_1 and we have :

$$-\Delta u_j > \lambda_j g \text{ in } \Omega_1.$$

Thus it follows that :

$$u_j > \lambda_j v \text{ in } \Omega_1,$$

where v is the positive solution of the problem :

$$\begin{cases} -\Delta v = g \text{ in } \Omega_1, \\ v = 0 \text{ on } \partial\Omega_1. \end{cases}$$

Thus for λ_j tending to ∞, u_j goes to ∞ uniformly over K. This fact together with (3.2), (3.3) gives :

$$\frac{f(u_j)}{u_j} \geq \mu_1 \text{ in } \Omega_0,$$

for some j large enough. Hence for this u_j,

$$\begin{cases} -\Delta u_j > \mu_1 u_j \text{ in } \Omega_0, \\ u_j > 0 \qquad \text{on } \partial\Omega_0, \end{cases}$$

which leads to a contradiction in view of the lemma. Thus the proposition follows. ∎

Now by the use of standard continuation arguments, essentially due to CRANDALL and RABINOWITZ, we can prove the following proposition (see also [5] C).

PROPOSITION 3.3 : There exists a number λ_1^*, $0 < \lambda_1^* < M$, such that there exists a branch of positive solutions $(\lambda, u_\lambda(x))$ for $0 < \lambda < \lambda_1^*$ satisfying :

(i) for $o < \alpha < 1$, u_λ is a C^1 map from $[o, \lambda_1^*)$ into $C^{2,\alpha}(\bar{\Omega})$.

(ii) for fixed $\lambda, u_\lambda(x)$ is the minimal positive solution,

(iii) $u_\lambda(x)$ is a non decreasing function of λ for $x \in \Omega$.

For the proof of this proposition, one can refer to [5] B, C and the references given there.

■

REMARK 3.4 : Define the operator $G : H_0^1 \times \mathbb{R} \to H^{-1}$ by :

(3.8) $$G(u,\lambda) = -\Delta u - f(u) - \lambda g(x),$$

then the partial derivatives are given by :

(3.9) $$\begin{cases} G_u(u,\lambda)v = -\Delta v - f'(u)\, v, \\ \\ G_\lambda(u,\lambda) = -g(x). \end{cases}$$

REMARK 3.5 : The eigenvalues μ_i of the linearized problem are given by $\mu_i = \mu_i(-f'(u),\Omega)$. By (ii) of proposition (3.3) and the fact :

(3.10) $$\mu_i(f,\Omega) < \mu_i(g,\Omega) \text{ for } f \prec g,$$

we see that the eigenvalues are decreasing along the branch of minimal positive solutions.

To describe the behaviour of the branch near λ_1^*, we need the following proposition on the boundedness of positive solutions.

PROPOSITION 3.6 : The positive solutions of (1.2) are bounded in $C^{2,\alpha}(\Omega)$ for $\lambda > 0$.

PROOF : For $N > 1$, with a few more assumptions on f, the proof follows easily from a theorem proved in [6]. So we indicate here a simple proof in one dimension, for the equation (1.1) under assumptions (3.1) to (3.4).

To obtain bounds in $C^1(0,1)$ for u, it is enough to show $\|u\|_{C^0}$ is bounded because we have, by integrating (1.1) in (0,1/2) after multiplying it by u'(x),

$$|\tfrac{1}{2}\, u'(0)|^2 \le |F(u(1/2))| + \lambda . \|g\|_{C^0}\, u(1/2), \tag{3.11}$$

where $F(s) = \int_0^s f(t)\, dt$. By the symmetry of the positive solutions, $\|u\|_{C^0} = u(1/2)$ and $u' > 0$ in (0,1/2) and $u' < 0$ in (1/2,1). (cf. for example, [11], section 2).

The proof is by contradiction and we suppose the existence of a sequence (u_j, λ_j) with $0 < \lambda_j < \lambda_1^*$, with $u_j(1/2) \to \infty$ as $j \to \infty$. By multiplying the equation (1.1) by u'(x) and integrating in (0,1/2) as before,

$$|\tfrac{1}{2}\, u'(0)|^2 \ge |F(u(1/2))| + \lambda .\{\min g\} . u(1/2).$$

This shows $u_j'(0)$ also goes to ∞, for j tending to ∞.

Fix some $\varepsilon > 0$, small and choose,

$$\sigma > \mu_1(0,(\varepsilon,1-\varepsilon)),$$

and m and ε_j such that :

$$\begin{cases} \dfrac{f(m)}{m} = \sigma, \\ u_j(\varepsilon_j) = m. \end{cases}$$

This is always possible because of the condition (3.3) on f.

For $0 \le u_j(x) \le m$, by integrating (1.1) between 0 and x,

$$|u_j'(x) - u_j'(0)| \le \int_0^x f(u_j(t))dt + \lambda \|g\|_{C^0}\, u_j(x) \le C_m.$$

This gives $u_j'(x) \to \infty$ for $0 \le u_j(x) \le m$. So there exists $D_j > 0$ such that,

$$\begin{cases} u_j'(x) > D_j \text{ for each } j, \\ D_j \to \infty \text{ as } j \to \infty. \end{cases}$$

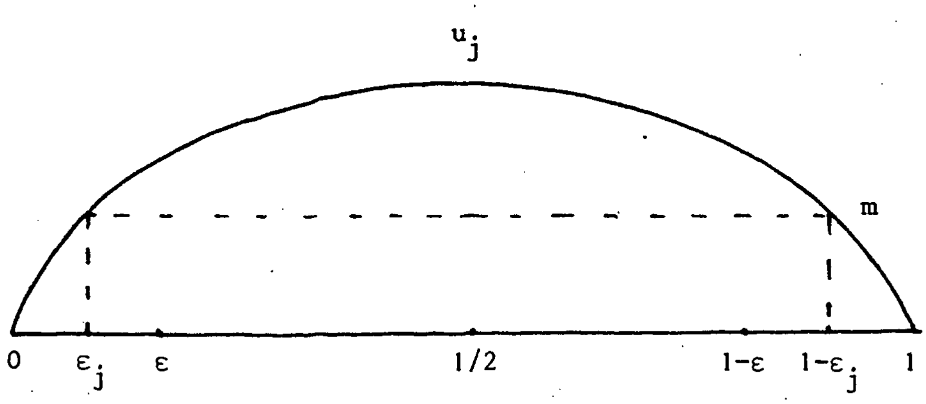


FIGURE 2

Then,

$$m = \int_0^{\varepsilon_j} u_j'(x)\, dx \geq D_j \varepsilon_j \implies \varepsilon_j \leq \frac{m}{D_j} \to 0.$$

Hence there exists J_0, large enough, such that for $j \geq J_0$, we have $\varepsilon_j < \varepsilon$. Then in $(\varepsilon_j, 1-\varepsilon_j)$,

$$\begin{cases} -u_j'' > \mu_1(0,(\varepsilon_j,1-\varepsilon_j))u_j, \\ u_j(\varepsilon_j) = u_j(1-\varepsilon_j) > 0, \end{cases}$$

because $\varepsilon_j < \varepsilon$ implies, $\mu_1(0,(\varepsilon,1-\varepsilon)) > \mu_1(0,(\varepsilon_j,1-\varepsilon_j))$. Now Lemma 3.1 gives the contradiction and so we get the boundedness of positive solutions. ∎

The following proposition describes the behaviour of the branch near λ_1^*, which indeed is a turning point.

<u>PROPOSITION 3.7</u> : For the equation (1.2), the minimal positive solution branch given by Proposition 3.3, is such that :

(i) $\lim_{\lambda \uparrow \lambda_1^*} u_\lambda = u^*$ exists and is the only positive solution at λ_1^*.

(ii) the positive solutions of (1.2) near λ_1^* form a smooth curve $(\lambda(s), u(s))$ for $|s| < \varepsilon$, with :

$$\begin{cases} (\lambda(0), u(0)) = (\lambda_1^*, u^*), \\ \lambda'(0) = 0 \text{ and } \lambda''(0) < 0. \end{cases}$$

<u>PROOF</u> : From Proposition 3.3, u_λ is left continuous at λ_1^*. This, along with the boundedness of positive solutions gives :

$$\lim_{\lambda \uparrow \lambda_1^*} u_\lambda = u^*.$$

Then, it is easy to verify that u^* is indeed a positive solution at λ_1^*. Also we observe that by the maximality of λ_1^*, the Freschet derivative G_u defined by (3.9) has to be singular at λ_1^* and hence,

$$\mu_1 \equiv \mu_1(-f'(u^*), \Omega) = 0. \tag{3.12}$$

Now to show that u^* is the only solution at λ_1^*, let us suppose that, if possible, there exists another positive solution v such that $v > u^*$. Then the non negative function $w = v - u^*$ satisfies,

$$-\Delta w = \frac{f(v) - f(u^*)}{v - u^*}(v - u^*) > f'(u^*)\,(w).$$

Hence, we have :

$$\begin{cases} -\Delta w - f'(u^*)w > 0 = \mu_1 \; w \text{ on } \Omega, \\ \qquad\qquad w = 0 \qquad\quad \text{on } \partial\Omega, \end{cases}$$

which leads to a contradiction, once again, by Lemma 3.1.

The proof of (ii) is an easy adaptation of the work of CRANDALL and RABINOWITZ (cf [5] B) to our case. Defining the operator $G(u,\lambda) : H_0^1 \times \mathbb{R} \to H^{-1}$ as in (3.8), we see that the partial derivatives are given by (3.9). At λ_1^*, $\mu_1 = 0$. This being the first eigenvalue, is of multiplicity 1. $G_u^0 \equiv G_u(\lambda_1^*,u^*)$ is then a self adjoint linear operator with :

(3.13) $$R(G_u^0) = \ker(G_u^0)^{\perp} = \{f \in H^{-1} \; ; \; \langle f,\phi\rangle_{H^{-1},H_0^1} = 0\},$$

where ϕ is the first eigen function corresponding to μ_1, which can be chosen to be positive in Ω. Also $G_\lambda^0 \equiv G_\lambda(\lambda_1^*,u^*)$ is not in $R(G_u^0)$ because,

$$\langle -g(x),\phi\rangle_{H^{-1},H_0^1} = -\int_\Omega g(x) \; \phi(x) \; dx \neq 0,$$

as both the functions do not change sign. Hence we have :

(i) dimension (ker G_u^0) = codimension ($R(G_u^0)$ = 1,

(ii) $G_\lambda^0 \notin R(G_u^0)$.

With these conditions, Theorem 3.2 in [5] B, gives us the following C^1 branch for $|s| < \varepsilon$,

(3.14) $$\begin{cases} \lambda(s) = \lambda_1^* + \tau(s), \\ u(s) = u^* + s\phi + z(s), \end{cases}$$

where,

$$\lambda(0) = \lambda_1^*, \; \lambda'(0) = 0,$$
$$u(0) = u^* \; , \; z(s) \in R(G_u^0), \; z'(0) = 0.$$

To determine the sign of $\lambda''(0)$, we differentiate the expression,

(3.15) $$G(\lambda_1^* + \tau(s), \; u^* + s\phi \; + z(s)) = 0,$$

twice with respect to s and evaluate it at s = 0, substituting,

$$\tau(0) = 0 \; ; \; z(0) = 0,$$

to get :

$$G_u^0 \; z''(0) = - [G_\lambda^0 \; \tau''(0) + G_{uu}^0 \; \phi^2].$$

Using Fredholm's alternative for the existence of $z''(0)$ we should have,

$$\int [g(x)\ \tau''(0) + f''(u^*)\ \phi^2]\ \phi = 0.$$

From this we deduce that :

$$\tau''(0) = - \frac{\int f''\ (u^*)\ \phi^3}{\int g(x)\ \phi}.$$

We have $f''(u^*) > 0$ because of (3.2) and $g(x) > 0$ by (3.4) and $\phi > 0$. Thus $\tau''(0) < 0$ and the proposition follows. ∎

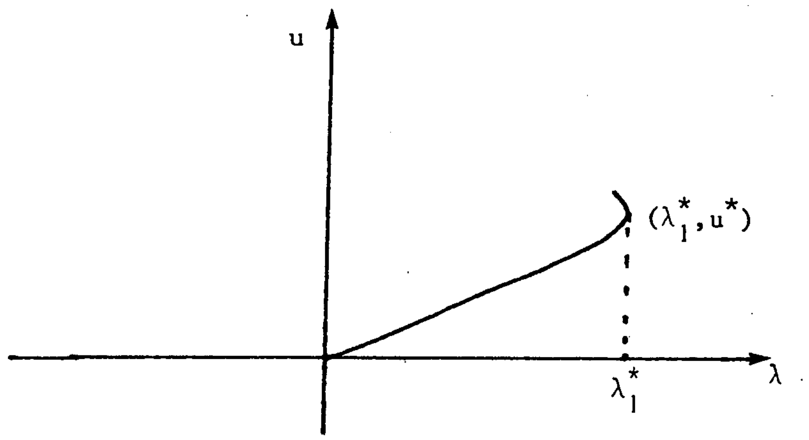


FIGURE 3

4 - POSITIVE SOLUTIONS IN ONE DIMENSION

Here we analyse the exact number of positive solutions for (1.1) when $\lambda > 0$ for $g(x) > 0$ and when $\lambda < 0$, only for constant g. Then we explore the boundedness of the positive solutions when $\lambda < 0$ and $g \equiv 1$.

PROPOSITION 4.1 : The equation (1.1) has :

(i) 2 positive solutions for $0 < \lambda < \lambda_1^*$,

(ii) at most one positive solution for $\lambda < 0$.

PROOF : First, we show that there exist at most two positive solutions for $\lambda \in (0,\lambda_1^*)$. By a direct integration of (1.1) after multiplying by u' between 0 and s, for $s \in (0,1/2)$, and comparison, we see that any two positive solutions of (1.1), have to be ordered (see, for example [12], Theorem 2.2).

Let there exist, if possible 3 positive solutions for some λ, say $u_1 < u_2 < u_3$. The non-negative differences $w_1 = u_2-u_1$ and $w_2 = u_3-u_2$ satisfy :

$$(4.1) \qquad \begin{cases} -w_1'' = \dfrac{f(u_2) - f(u_1)}{u_2-u_1} \quad w_1 \equiv h(u_2,u_1)w_1, \\ -w_2'' = \dfrac{f(u_3) - f(u_2)}{u_3-u_2} \quad w_2 = h(u_3,u_2)w_2. \end{cases}$$

By multiplying the first by w_2 and the second by w_1, and subtracting after integration by parts,

$$0 = \int_0^1 \{h(u_2,u_1) - h(u_3,u_2)\}\ w_1.w_2.$$

By convexity of f,

$$h(u_2,u_1) < h(u_3,u_2),$$

and hence none of the functions in the integrand changes sign which leads to a contradiction. Thus there can exist at most 2 positive solutions to (1.1).

Now to show the existence of at least two positive solutions, we can employ a standard argument using degree theory. (refer for example, [14]). Thus (i) is proved.

Supposing that there exist 2 positive solutions u_1, u_2 for $\lambda = -\mu$, $\mu > 0$, we can show as before that they have to be ordered, say $u_1 > u_2$. Let $w = u_1 - u_2$, which is a non negative function.

$$(4.2) \qquad \begin{cases} -w'' = \dfrac{f(u_1) - f(u_2)}{u_1 - u_2}\, w > f'(u_2)\, w, \\ -u_2'' = f(u_2) - \mu,\ \mu > 0. \end{cases}$$

Multiplying these 2 equations by u_2 and w respectively and integrating any one of them by parts and subtracting,

$$0 > \int (f'(u_2)u_2 - f(u_2))\, w + \mu \int w.$$

As f is convex, the function $\{f'(s).s - f(s)\}$ is non-negative. Thus both the integrals have to be positive. This contradiction shows that we can have at most one positive solution for $\lambda < 0$, for (1.1). ∎

The next proposition shows that the positive solution branch cannot go to $-\infty$, in λ, at least when $g \equiv 1$.

PROPOSITION 4.2 : There exists a negative constant $-M_1$, such that for $\lambda < -M_1$, there are no positive solutions to the equation (1.1) with $g(x) \equiv 1$.

PROOF : For convenience, we replace λ by $-\lambda$ in the following. Let us suppose that there exist a positive solution to the equation,

$$(4.3) \qquad \begin{cases} -u'' = f(u) - \lambda \text{ in } (0,1),\ \lambda > 0, \\ u(0) = u(1) = 0, \end{cases}$$

for large values of λ. As $u'' > 0$ at 0 and 1 and $u''(1/2) < 0$, u has to be convex near the extremities and concave in the middle. Let a be the point in $(0,1/2)$ where $u''(a) = 0$. Hence at a,

$$(4.4) \qquad f(u(a)) = \lambda.$$

Now choose a point $b \in (0,a)$ such that $u''(b) = \frac{\lambda}{2}$. At b,

$$(4.5) \qquad f(u(b)) = \frac{\lambda}{2}.$$

The idea is to prove that the point b tends to 0 and so does the point a, for λ large.

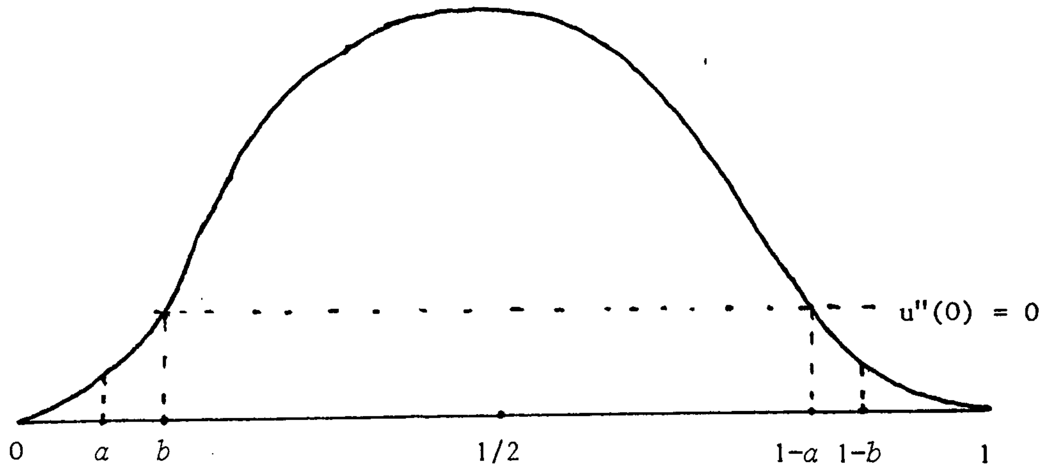


FIGURE 4

By integrating u'' between 0 and b,

(4.6) $$b.\frac{\lambda}{2} < \int_0^b u''\,dx = u'(b) - u'(0).$$

Multiplying (4.3) by u' and integrating between 0 and b,

(4.7) $$\begin{cases} -\frac{1}{2}[u'^2(b) - u'^2(0)] = F(u(b)) - \lambda u(b), \\ u'^2(b) - u'^2(0) < 4f(u(b)).u(b) - 2f(u(b)) < 4f(u(b)).u(b). \end{cases}$$

Combining (4.6) and (4.7), we have (note that $u'(b)-u'(0) < \sqrt{u'^2(b)-u'^2(0)}$)

$$b < \frac{2\sqrt{f(u(b)).u(b)}}{f(u(b))} = 2\sqrt{\frac{u(b)}{f(u(b))}}\ .$$

As $u(b) = f^{-1}(\frac{\lambda}{2}) \to \infty$ as $\lambda \to \infty$, using the condition (3.3), the right hand side tends to 0. The function f and u both being convex in $(0,a)$, and u being monotone, the composition fou is also convex and hence,

$$(fou)(\tfrac{a}{2}) < \frac{1}{2}\,(fou)(a) = (fou)(b)\ ,$$

which implies, by the monotonicity of (fou) in (0,1/2),

$$a < 2b.$$

Thus we can assume that for λ large enough,

(4.8) $$a < \varepsilon,$$

for a given ε. Let $\mu_1 = \mu_1(0,(\varepsilon,1-\varepsilon))$. Define :

$$v(x) = u(x) - u(a) \quad \text{in } (a,1-a).$$

Then we see that by (4.3) and the convexity of f, (in particular $f'(s) > \frac{f(s)}{s}$) :

$$-v'' > v \left\{\frac{f(u(a))}{u(a)}\right\}.$$

Now choose λ large enough so that :

$$\frac{f(u(a))}{u(a)} > \mu_1,$$

and also (4.8) holds. Then we have,

$$\begin{cases} -v'' > \mu_1 v \text{ in } (a,1-a), \\ v \geq 0,\ v(a) = v(1-a) = 0, \end{cases}$$

which is impossible by Lemma 3.1. Thus equation (4.3) does not possess a positive solution for arbitrarily large λ. ∎

REMARK 4.3 : Using Proposition 4.2, we can extend the proof of Proposition 3.5, for the case $\lambda < 0$, for the equation (1.1), when $g(x) \equiv 1$.

Fix some $\varepsilon > 0$ and let $\mu_1^\varepsilon = \mu_1(0,(\varepsilon,1-\varepsilon))$. Choose :

$$\sigma > \{\mu_1^\varepsilon + M_1\}, \tag{4.9}$$

where M_1 is the constant supplied by Proposition 4.2, and then choose m such that :

$$\frac{f(m)}{m} = \sigma. \tag{4.10}$$

If $m \leq 1$, we can still increase σ and choose m such that $m > 1$ and (4.10) is also satified. This must be possible because of the condition (3.3) on f.

Now we choose w, the first eigenfunction corresponding to μ_1^ε such that,

$$\begin{cases} \text{(i)} \quad w \geq 0, \\ \text{(ii)} \quad \int_\varepsilon^{1-\varepsilon} w = 1. \end{cases} \tag{4.11}$$

As before, let $u(\varepsilon_j) = m$ with $\varepsilon_j < \varepsilon$. Using these, we have :

$$f(u) - \lambda = \frac{f(u)}{u}.u - \lambda > \frac{f(m)}{m}.u - \lambda > \sigma u - M_1.$$

Thus we have :

$$\left.\begin{aligned} -u'' &> \sigma u - M_1, \\ -w'' &= \mu_1^\varepsilon.w, \end{aligned}\right\} \text{ in } (\varepsilon,1-\varepsilon).$$

As before, using these two equations :

$$0 > (\sigma-\mu_1^{\varepsilon})\int u.w - M_1\int w > (\sigma-\mu_1^{\varepsilon})m\int_{\varepsilon}^{1-\varepsilon} w - M_1,$$

$$0 > (\sigma-\mu_1^{\varepsilon}) - M_1,$$

which contradicts our choice of σ.

Thus,we see that the positive solutions for $\lambda > 0$ as well as $\lambda < 0$ are bounded, and hence, the branch S_1 is bounded in the space X. ∎

5 - FIRST BIFURCATION POINT

From now on, we restrict our attention to the equation (1.1) for the case :

(5.1) $$g(x) \equiv 1.$$

With this additional assumption,we can analyse more precisely the behaviour of the positive branch for negative λ and study the bifurcation phenomenon.

From earlier proposition, we know that the positive solution branch turns back at λ_1^*. In this upper branch for $\lambda \in (0,\lambda_1^*)$ we can show that (cf. proof of Proposition (5.1), (i), below) :

(5.2) $$\mu_1 \equiv \mu_1(-f'(u),(0,1)) < 0 < \mu_2(-f'(u),(0,1)) \equiv \mu_2.$$

Thus there are no singular points till 0 and again at 0, G_u being invertible, by implicit function theorem, the branch extends to negative λ, till it encounters a singular point, say λ_{-1}^*. If μ_1 is 0 here, then it has to be a turning point but by the uniqueness of positive solutions for negative λ, this is impossible. So it has to be μ_2 which becomes 0. The following proposition describes what happens at λ_{-1}^*.

PROPOSITION 5.1 : At λ_{-1}^* (denoted hereafter by λ_0) where the second eigenvalue μ_2 is 0,

(i) the second eigenfunction corresponding to μ_2 is given by u_0',i.e. $\frac{du_0}{dx}$; (u_0 is the positive solution of (1.5) at $\lambda = \lambda_0$) ;

(ii) the bifurcation equation is non-degenerate and has 2 distinct tangent directions as its roots ;

(iii) the positive branch after λ_0, can be continued as S_3^-. Further there exits a second continuous branch of solutions which are not symmetric with respect to 1/2 and hence distinct from the first branch. This consists of S_2^+ and S_2^-.

PROOF : At λ_0, by assumption there exists ϕ_2, odd with respect to 1/2, such that :

(5.3) $$\begin{cases} -\phi_2'' = f'(u_0)\,\phi_2 & \text{in } (0,1), \\ \phi_2(0) = \phi_2(1) = 0. \end{cases}$$

Note that u_0' has only one zero between 0 and 1. Extend u_0 in a C_1 fashion to $\hat{u}_0$ in (α,β) for $\alpha < 0$, $\beta > 0$ such that :

(i) α is the first zero of $\hat{u}_0'$, to the left of 0.

(ii) β is the first zero of $\hat{u}_0'$, to the right of 1.

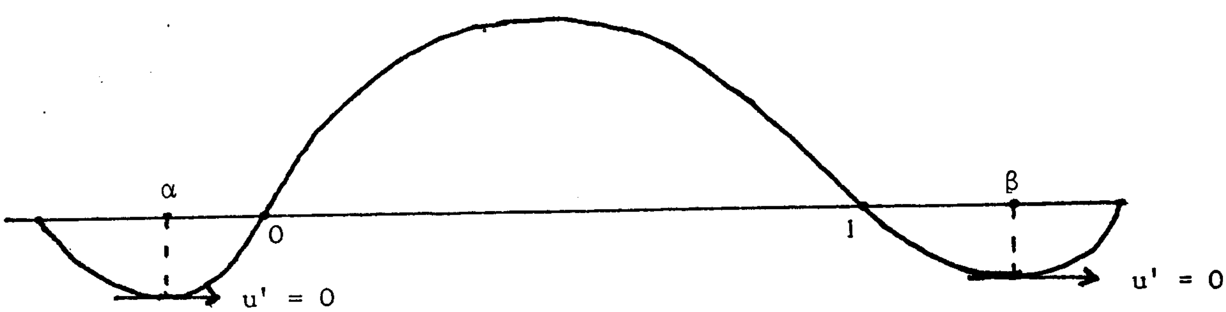


FIGURE 5

One way of doing this, is to solve the initial value problems,

$$\begin{cases} -u'' = f(u) + \lambda_0 & \text{in } (-\infty,0), \\ u(0) = 0 \;;\; u'(0) = u_0'(0), \end{cases}$$

and,

$$\begin{cases} -u'' = f(u) + \lambda_0 & \text{in } (1,\infty), \\ u(1) = 0 \;;\; u'(1) = u_0'(1), \end{cases}$$

and then to choose α and β suitably as above.

Then $\hat{u}_0'$ satisfies the equation,

$$\begin{cases} -(\hat{u}_0')'' = f'(\hat{u}_0).\hat{u}_0' & \text{in } (\alpha,\beta), \\ \hat{u}_0'(\alpha) = \hat{u}_0'(\beta) = 0, \end{cases}$$

and $\hat{u}_0'$ has only one zero in (α,β). Hence,

$$\mu_2(-f'(\hat{u}_0),(\alpha,\beta)) = 0.$$

But from (5.3),

$$\mu_2(-f'(u_0),(0,1)) = 0.$$

As $(0,1) \subset (\alpha,\beta)$ we have,

$$\mu_2(-f'(u_0),(0,1)) > \mu_2(-f'(\hat{u}_0),(\alpha,\beta)).$$

This leads to a contradiction unless $\alpha = 0$ and $\beta = 1$.

That is to say $u_0'(0) = u_0'(1)$ and hence u_0' itself is the second eigenfunction (see Figure 6 for the shape of u_0).

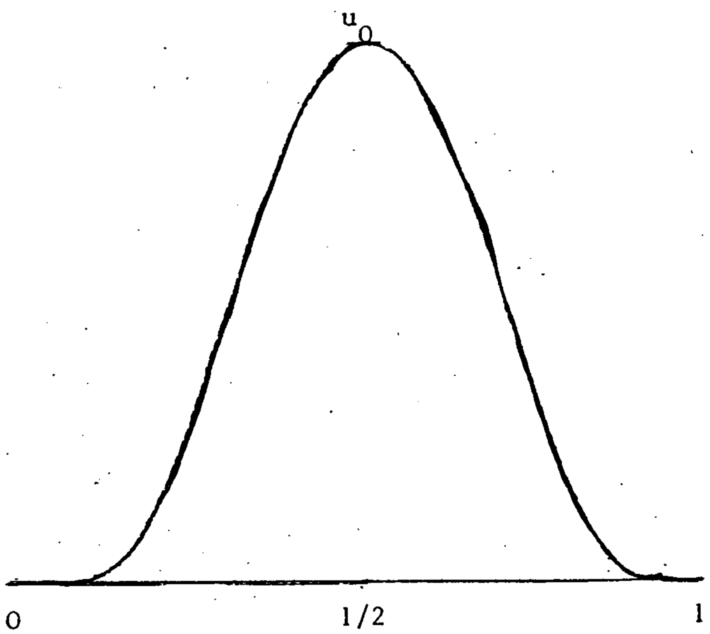


FIGURE 6

Taking the operator G to be from C_0^2 to C_0, defined as in (3.8), we have :

(5.4) $$\begin{cases} G_u(u,\lambda)(v) = -v'' - f'(u)v, \\ G_\lambda(u,\lambda) = -1. \end{cases}$$

Then we can check,

$$\begin{cases} R(G_u^0) = \{w \in C_0 \;/\; \int_0^1 w\,\phi_2 = 0\}, \\ \text{codimension } R(G_u^0) = 1 = \text{dimension } \ker(G_u^0). \end{cases}$$

As ϕ_2 is antisymmetric with respect to 1/2, we see that :

(5.5) $$G_\lambda^0 \in R(G_u^0).$$

Now we use a standard argument (cf [11]A, section 5) to derive the bifurcation equations. Assuming that the branch is given by $(u(s),\lambda(s))$, we have,

$$G(u(s),\lambda(s)) = 0 \text{ for } s \in (s_0-\varepsilon, s_0+\varepsilon),$$

where $(u(s_0),\lambda(s_0)) = (u_0,\lambda_0)$. Differentiating with respect to s,

(5.6) $$G_u(u(s),\lambda(s))u'(s) + G_\lambda(u(s),\lambda(s))\,\lambda'(s) = 0.$$

At s_0, this reduces to ,

(5.7) $$G_u^0.u'(s) + G_\lambda^0.\lambda'(s) = 0.$$

Let ϕ be the solution of the problem,

(5.8) $$G_u^0\,\phi + G_\lambda^0 = 0 \;;\; \int_0^1 \phi\,\phi_2 = 0.$$

We note that ϕ is unique because of the orthogonality condition.
From (5.7) and (5.8),

$$u'(s) - \lambda'(s)\phi = \alpha_1\phi_2,$$

for some scalar α_1. Let,

$$\begin{cases} \lambda'(s) = \alpha_0, \\ u'(s) = \alpha_0\phi + \alpha_1\phi_2. \end{cases}$$

Now to get the bifurcation equation, giving two sets of values for (α_0,α_1), we differentiate (5.6) once again, to get :

$$G_{uu}(u'(s))^2 + G_{u\lambda}u'(s).\lambda'(s) + G_u u''(s) + G_{u\lambda}u'(s).\lambda'(s) + G_{\lambda\lambda}(\lambda'(s))^2 + G_\lambda\lambda''(s) = 0.$$

At $s = s_0$,

$$G_u^0(u''(s) = -[G_{uu}^0(u'(s))^2 + 2G_{u\lambda}^0(u'(s),\lambda'(s)) + G_{\lambda\lambda}^0(\lambda'(s))^2] - G_\lambda^0.\lambda''(s).$$

The bracketted term is in $R(G_u^0)$ as the other two do. So,

$$\int_0^1 \phi_2[---] = 0,$$

which yields after simplification,

$$(5.9) \qquad \begin{cases} a\alpha_0^2 + 2b\alpha_0\alpha_1 + c\alpha_1^2 = 0 ; \\ a = \int\phi_2\ G_{uu}^0(\phi_2,\phi_2) = -\int f''(u_0)\phi_2^3 ; \\ b = \int\phi_2\ G_{uu}^0(\phi_2,\phi) + \phi_2\ G_{\lambda u}^0\phi = -\int f''(u_0)\phi_2^2\ \phi ; \\ c = \int\phi_2\ G_{uu}^0(\phi,\phi) + 2\phi_2\ G_{\lambda u}^0\ \phi + G_{\lambda\lambda}^0\ \phi_2 = -\int f''(u_0)\phi_2\phi^2. \end{cases}$$

Because of the antisymmetry of ϕ_2 and the symmetry of the other functions,

$$a = 0 ; c = 0.$$

If $b \neq 0$, then the roots of the equation (5.9) are (1,0) and (0,1), so that the tangents to the curve $(u(s),\lambda(s))$ are given by,

$$(5.10) \qquad (du,d\lambda) = (\phi,1) \text{ or } (du,d\lambda) = (\phi_2,0).$$

Now we show that $b \neq 0$. Writing b as,

$$b = -\int f''(u_0).(u_0')^2\ \phi = -\int(f'(u_0))'(u_0'\phi),$$

we integrate by parts and use the equation for u_0 to get,

$$b = \int_0^1\{f(u_0) - \lambda_0\ f'(u_0)\phi\}.$$

Now using the equation for u_0 and ϕ,

$$(5.11) \qquad b = \int_0^1 -u_0'' + \lambda_0\phi'' = \lambda_0 \int_0^1 \phi'' = -2\lambda_0\ \phi'(0)$$

as ϕ is symmetric (cf. Remark 5.2 for the details).

Next we show that either ϕ is completely negative or has to change sign at least twice (see Figure 7 and Figure 8).

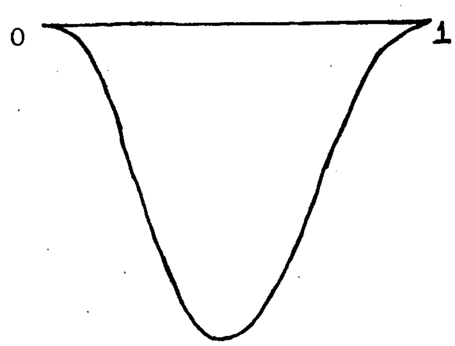


FIGURE 7

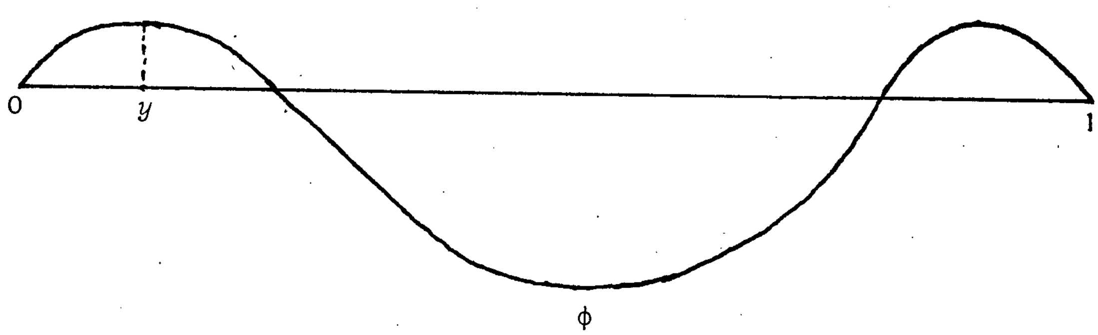


FIGURE 8

Let ϕ_1, chosen to be positive, be the first eigenfunction of the linearized problem :

$$\begin{cases} -\phi_1'' = f'(u_0)\phi_1 + \mu_1\phi_1 & \text{in } (0,1), \\ \phi_1(0) = \phi_1(1) = 0. \end{cases}$$

Rewriting equation (5.8) for ϕ, we have :

$$\begin{cases} -\phi'' = f'(u_0)\phi + 1 & \text{in } (0,1), \\ \phi(0) = \phi(1) = 0 \ ; \int_0^1 \phi\ \phi_2 = 0. \end{cases} \tag{5.12}$$

As before, comparing these 2 equations,

$$0 = -\int_0^1 \phi_1 + \mu_1 \int_0^1 \phi\, \phi_1.$$

Since $\mu_1 < 0$, this equation would be impossible if ϕ were completely positive. So either ϕ changes sign or is completely negative. As ϕ is symmetric, it has to have even number of zeroes, say 2k, for $k \geq 1$, if at all ϕ changes sign. We now show that k has to be 1.

Defining the wronskian of u_0' and ϕ,

$$W(x) = \begin{vmatrix} u_0' & u_0'' \\ \phi & \phi' \end{vmatrix},$$

we see that :

(5.13) $\qquad W'(x) = -u_0'(x) \qquad$ and $\qquad W(x) = -u_0(x).$

At a point $x \in (0,1/2)$, where $\phi(x) = 0$,

$$\phi'(x) = -\frac{u_0(x)}{u_0'(x)} < 0.$$

So ϕ can cross zero only from positive values to negative values in $(0,1/2)$ and thus cannot have more than 2 zeroes. Hence $k = 1$ (see Figure 9).

Now we show that ϕ cannot be completely negative. If ϕ were completely negative, then $\phi'(0) \leq 0$ and hence ,

$$-2\lambda_0\phi'(0) \leq 0.$$

Using (5.11) and the definition of b in (5.9), we conclude that :

$$\int_0^1\{-f''(u_0)(u_0')^2\phi\} \leq 0,$$

which is impossible because the integrand would in fact be positive. Thus the only possibility for ϕ, is to change sign twice, starting from positive values (see Figure 9). Let y be the first zero of ϕ'. Integrating the equation (5.12) between 0 and y we get :

$$\phi'(0) = \int_0^y f'(u_0)\phi + y > 0.$$

Now using (5.11) we obtain $b \neq 0$ and (ii) of Proposition (5.1) is proved.

Thus we have at (u_0,λ_0),

(i) G_u^0 is singular and dimension(ker G_u^0) = 1, codimension $(R(G_u^0)) = 1$,

(ii) $G_\lambda^0 \in R\,(G_u^0)$,

(iii) the non degeneracy condition $b \neq 0$ holds.

With these, we can use the standard arguments (cf [5] A, Theorem 1.7) to show that (λ_0,u_0) is indeed a bifurcation point and the distinct curves which cut at (λ_0,u_0), having tangent directions given by (5.10), are :

$$\text{(5.14)} \qquad \begin{cases} \lambda(t) = \lambda_0 + A_1 t + O(t^2), \\ u(t) = u_0 + \phi . t + O(t^2) \ ; \end{cases}$$

and,

$$\text{(5.15)} \qquad \begin{cases} \lambda(t) = \lambda_0 + A_0 t^2 + O(t^3), \\ u(t) = u_0 + \phi_2 t + O(t^2). \end{cases}$$

As $\phi'(0) = \frac{d}{d\lambda}(u'(0)) > 0$, we have for λ decreasing further from λ_0, $u'(0)$ decreasing. Thus $u'(0) < 0$ after λ_0, for the branch continued from S_1^+ and hence by the symmetry of the solutions, it must be S_3^-. This is the one represented in (5.14). The expression (5.15) for the other branch suggests that it is S_2. The two branches, namely $S_1^+ \cup S_3^-$ and $S_2^+ \cup S_2^-$ are indicated in Figures 9 and 10 respectively. ∎

First branch given by (5.14)

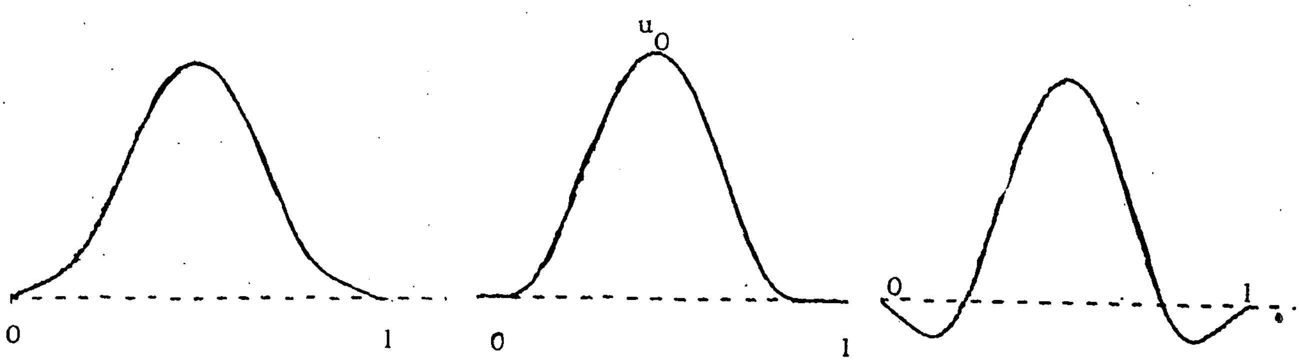


FIGURE 9

Second branch given by (5.15)

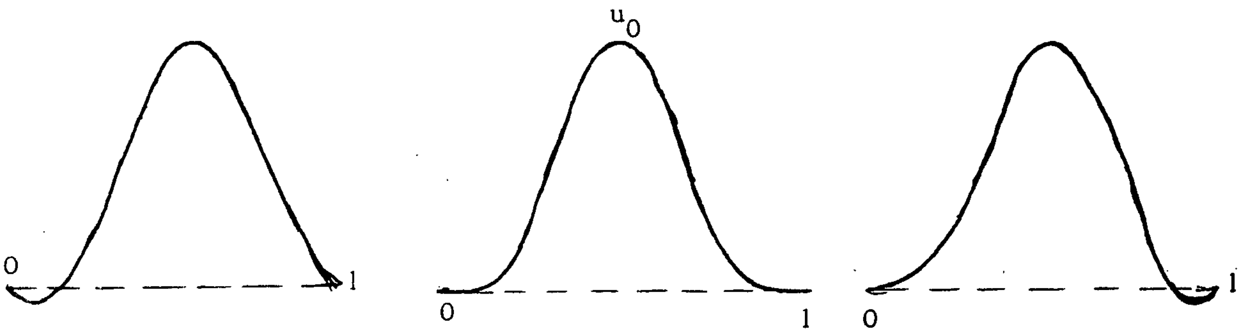


FIGURE 10

REMARK 5.2 : Here we show in detail the derivation of the two equivalent expressions for the term b.

$$b = -\int_0^1 f''(u_0)\phi_2^2\ \phi.$$

Using the fact $\phi_2 = u_0'$ and integrating by parts,

$$\begin{aligned} -\ b &= \int_0^1 f''(u_0)(u_0')^2\phi = \int_0^1 (f'(u_0))'(u_0'\phi) \\ &= -\int_0^1 f'(u_0)(u_0'\phi)' = -\int_0^1 f'(u_0)[u_0''\phi + u_0'\phi']. \end{aligned}$$

The use of the equation for u_0 results in further simplifications :

$$\begin{aligned} -\ b &= \int_0^1 f'(u_0)(f(u_0)+\lambda)\phi - \int_0^1 (f(u_0))'\phi' \\ &= \int_0^1 \{f'(u_0)\ f(u_0)\phi + \lambda f'(u_0)\phi\} + \int_0^1 f(u_0)\phi'' \\ &= \int_0^1 \{f'(u_0)\ f(u_0)\phi + \lambda f'(u_0)\phi\} - \int_0^1 f(u_0)\{f'(u_0)\phi + 1\} \\ &= \int_0^1 \{\lambda f'(u_0)\phi - f(u_0)\}. \end{aligned}$$

Multiplying the equation (5.12) by λ and subtracting it from the equation for u_0, we get,

$$\int_0^1 -u_0'' + \lambda\phi'' = \int_0^1 \{f(u_0) - \lambda f'(u_0)\phi\},$$

which results in :

$$-2\lambda\phi'(0) = \int_0^1 \{f(u_0) - \lambda f'(u_0)\phi\},$$

because of the symmetry of the function ϕ and the fact $u_0'(0) = u_1'(0) = 0$. From this, we obtain :

$$b = \int_0^1 \{f(u_0) - \lambda f'(u_0)\phi\} = -2\lambda\phi'(0). \qquad \blacksquare$$

REMARK 5.3 : An argument similar to the one used in the proof of (i) of Proposition 5.1 gives the following result : we have along $\mathrm{Int}(S_k)$, for $k \geq 2$,

(5.16) $\qquad \mu_{k-1}(-f'(u),(0,1)) < 0 < \mu_{k+1}(-f'(u),(0,1)).$

In fact we have all along $\mathrm{Int}(S_k)$,

$$0 < \mu_{k+1}(-f'(u),(0,1)),$$

because for any $u \in S_k$, $v = \hat{u}'$, extended in $[\alpha,\beta]$ as in the proof of Proposition 5.1, has k nodes in (α,β) satisfies the linearized equation and hence,

$$0 = \mu_{k+1}(-f'(u),(\alpha,\beta)) < \mu_{k+1}(-f'(u),(0,1)).$$

If $k \geq 2$, u' has at least 2 nodes in $(0,1)$. Choosing the last 2 zeroes closest to 0 and 1 and of u' as α_1 and β_1, we have :

$$0 = \mu_{k-1}(-f'(u),(\alpha_1,\beta_1)) > \mu_{k-1}(-f'(u),(0,1)),$$

because u' has $(k-2)$ zeroes in (α_1,β_1). Thus we obtain the relation (5.16). $\blacksquare$

6 - GLOBAL BEHAVIOR OF S_2 AND OTHER S_{2k}, $k > 1$

In the following 2 sections, in addition to the condition (5.1) on g we need to have also the following restriction on f :

(6.1) $$f(u) = |u|^{p-1}u, \ p > 1.$$

This condition helps us to define a mapping from S_1 to other S_n for $n > 1$ and thus enables us to study the global behaviour of S_n, $\forall$ n. First we note that Remark 4.3 can be extended for the branch S_n, $n > 1$ also.

Observe that for any S_n, there exists λ_n^- and λ_n^+ with,

$$\lambda_n^- < 0 < \lambda_n^+,$$

such that there exists no solution $u \in S_n$ for $\lambda > \lambda_n^+$ and for $\lambda < \lambda_n^-$. Let $u \in S_n$ with (n-1) nodes at $\{a_j\}_{j=1}^{n-1}$. There exists at least one interval (a_j, a_{j+1}) for which ,

$$(a_{j+1}-a_j) \geq \frac{1}{n}.$$

Defining,

$$v(y) = (a_{j+1}-a_j)^{2/(p-1)} u\{(a_{j+1}-a_j)y + a_j\} \text{ for } y \in (0,1),$$

we see that v is a positive or negative solution for (1.5) at,

$$\lambda' = (a_{j+1}-a_j)^{2p/(p-1)} \lambda.$$

But λ' has to lie in $[\lambda_{-1}^*, \lambda_1^*]$ which implies,

(6.2) $$n^{2p/(p-1)} \lambda_{-1}^* \leq \lambda \leq n^{2p/(p-1)} \lambda_1^*.$$

Thus we get the constants λ_n^- and λ_n^+.

Now using this fact, we can prove that the solutions $u \in S_n$ are bounded in $C_0^1(0,1)$, on the same lines as in the proof of Proposition 3.5. Thus part (i) of Theorem 1.1 is proved.

Now we analyse the behaviour of S_2. As the solution $u_0^+ \in S_2^+$, at $\lambda = 0$ is non-singular, (refer Lemme 8 of [4]), we can extend the branch S_2^+, (S_2^- from $u_0^- \in S_2^-$), in a small neighbourhood of $\lambda = 0$. Next we observe that a solution $(\lambda, u) \in S_2^+$, corresponds to the solutions $(-\lambda, u_1) \in S_2^+$, $(-\lambda, u_2) \in S_2^-$ and $(\lambda, u_3) \in S_2^-$, where :

$$\begin{cases} u_1(x) = -u(1-x), \\ u_2(x) = -u(x), \\ u_3(x) = u(1-x), \end{cases}$$

because of the restriction (6.1) on f. Thus the behaviour of the branch S_2 in one quadrant determines the behaviour in all the other three.

From part (iii) of Proposition (5.1), we see that S_2^+ and S_2^- meet at the bifurcation point (λ_0, u_0). Analysing the expression (5.15) further, we see that :

$$A_0 < 0.$$

(cf. Remark 6.1, for the computation of A_0). Hence the branch S_2 bends forward at (λ_0, u_0), as in Figure 11.

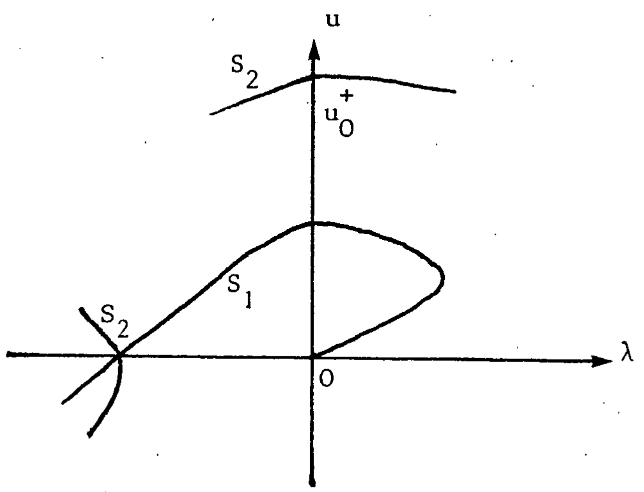


FIGURE 11

Now we show that S_2 is a smooth curve from $\lambda = 0$ to $\lambda = \lambda_0$. This we do by fabricating the solution $\hat{u} \in S_2$ from the solution $u \in S_1$ for $\lambda \in (\lambda_0, 0]$. Then we show that this branch S_2 does not bifurcate between λ_0 and 0.

Let $u(x)$ be the positive solution at λ, where $\lambda_0 < \lambda \le 0$. Extend u in a C^1 fashion, up to its second zero, say $a(\lambda)$ as was done in the proof of (i) of Proposition (5.1).

$u(x)$ in S_1 for $x \in (0,a(\lambda))$ goes to $\hat{u}(y)$ for $y \in (0,1)$ under the following map :

$$\hat{u}(y) = (a(\lambda))^{2/(p-1)} \; u\,(a(\lambda)\; y),\; y \in (0,1).$$

Then $\hat{u}$ is a solution of the equation at $\hat{\lambda} = (a(\lambda))^{2p/(p-1)}\;\lambda$, and $\hat{u}$ has just one node in $(0,1)$. For $\lambda = 0$, $a(\lambda) = 2$ and for $\lambda \to \lambda_0$, $a(\lambda) \to 1$.

Thus if we define $h : C_0^2(0,1) \times (\lambda_0,0] \to C_0^2(0,1) \times (\lambda_0,0]$ by the following :

$$h(u,\lambda) = (h_1(u,\lambda),h_2(u,\lambda)) = (\hat{u},\hat{\lambda}),$$

where,

$$h_1(u,\lambda) = \hat{u}(x) = (a(\lambda))^{2/(p-1)}\;\tilde{u}\;(a(\lambda)\;x),$$

$$h_2(u,\lambda) = \hat{\lambda} = (a(\lambda))^{2p/(p-1)}\;\lambda,$$

then $G(u,\lambda) = 0 = G(\hat{u},\hat{\lambda})$ and we have $(u,\lambda) \in S_1$ for $\lambda \in (\lambda_0,0]$ and $(\hat{u},\hat{\lambda}) \in S_2$ for $\lambda \in (\lambda_0,0]$. Thus we get a smooth branch, a part of S_2 between 0 and λ_0. As $a^{2p/(p-1)}\;\lambda < \lambda$ for $\lambda < 0$, and as $\hat{\lambda} \to \lambda$ at λ_0, this branch S_2 turns back at least once.

Suppose some point $(\hat{u},\hat{\lambda})$ in this branch S_2 is singular, i.e. G_u is singular. We note that it can only be the second eigenvalue which becomes 0 here (cf. Remark 5.3) because,

$$\mu_1(-f'(u),(0,1)) < 0 < \mu_3(-f'(u),(0,1)),$$

along Int(S_2). Hence there exists a function ϕ_2 with one node,satisfying :

$$(6.3) \qquad \begin{cases} -\phi_2'' = f'(\hat{u})\;\phi_2 & \text{in } (0,1), \\ \phi_2(0) = \phi_2(1) = 0. \end{cases}$$

Multiplying the equation :

$$(6.4) \qquad \begin{cases} -\hat{u}'' = f(\hat{u}) + \hat{\lambda} & \text{in } (0,1), \\ \hat{u}(0) = \hat{u}(1) = 0, \end{cases}$$

by $(x\phi_2')$ and integrating by parts (cf. Remark 6.2 for details) we get :

$$(6.5) \qquad \frac{2p}{p-1}\;\hat{\lambda}\int_0^1 \phi_2 = \hat{u}'(1)\;\phi_2'(1).$$

If $\int_0^1 \phi_2$ were 0, then $\hat{u}'(1)$ would have to be 0 and hence the corresponding $u'(1)$ for $u \in S_1$. But for λ, in $(\lambda_0,0]$, $u'(1)$ is not 0 and hence $\int_0^1 \phi_2 \neq 0$. Following our earlier notation,this condition means,

$$G_\lambda(\hat{u},\hat{\lambda}) \notin R(G_u(\hat{u},\hat{\lambda})).$$

Then we can show that $(\hat{u},\hat{\lambda})$ is indeed a turning point (refer to the proof of Proposition 3.7). Thus, a singular point $(\hat{u},\hat{\lambda})$ for $\lambda_0 < \lambda \leq 0$, if any, can only be a turning point but cannot be a bifurcation point.

At this juncture, as it is not clear whether the maximum number of solutions in S_2^+ at any λ is two or more, we are not able to conclude that there is just one turning point. For $f(u) = u^3$, however, the numerical evidence shows that there is just one turning point. Thus in any case, the branch S_2 is a closed C^1 curve and we denote by R_2 the last turning point of S_2.

Next we note that there is one-one correspondence between S_2 and S_{2n}, $n \geq 1$. We can get S_{2n} from S_2, by extending a solution in S_2 up to $(0,n)$ or by patching together n copies of this solution and vice versa. Hence each S_{2n}, $n \geq 1$, has also to be closed. This completes the proof of (iii) of Theorem (1.1). The last turning point of $S_{2n}^{\pm}$ is denoted by $R_{2n}^{\pm}$.

The following remark indicates the general method of calculation of the coefficient A_0, occuring in the expression (5.15).

<u>REMARK 6.1</u> : Our aim here is to solve :

$$G(\lambda,u) = 0, \tag{6.6}$$

in a neighbourhood of the bifurcation point (λ_0,u_0), where $G : \mathbb{R} \times V \to V$, for a Banch space V. Following our earlier notation, ϕ_2 is the second eigenfunction of the linearized operator G_u^0. Let $\phi_2^* \in V^*$ satisfy :

$$\begin{cases} G_u^{0*}\, \phi_2^* = 0, \quad (G_u^{0*} = \text{adjoint of } G_u^0), \\ \langle \phi_2^*,\phi_2 \rangle = 1. \end{cases}$$

Then we have,

$$V_1 = \ker(G_u^0) = \operatorname{span}\{\phi_2\},$$

$$V_2 = R(G_u^0) = \{v \in V \; ; \; \langle v,\phi_2^* \rangle = 0\}.$$

In our case,

$$V = V_1 \oplus V_2,$$

and $G_u^0 \Big|_{V_2}$ is an isomorphism . We define,

$$L = G_u^0 \Big|_{V_2}^{-1},$$

Q = projection operator from V_2 into V.

Solving (6.6) amounts to solving,

$$(6.7) \qquad \begin{cases} QG(\lambda,u) = 0 \text{ in } V_2, \\ (I-Q)G(\lambda,u) = 0 \text{ in } V_1, \end{cases}$$

or equivalently,

$$(6.8) \qquad \begin{cases} QG(\lambda,u) = 0 \text{ in } V_2, \\ \langle G(\lambda,u),\phi_2^*\rangle\phi_2 = 0 \text{ in } V_1. \end{cases}$$

Applying implicit function theorem to the first equation, we get a function $V(\xi,\alpha) : \mathbb{R}^2 \to V_2$, defined in a neighbourhood $(-\xi_0,\xi_0) \times (-\alpha_0,\alpha_0)$ of $(0,0)$, satisfying,

$$(6.9) \qquad \begin{cases} QG(\lambda_0 + \xi,\ u_0 + \alpha\phi_0 + v(\xi,\alpha)) = 0, \\ v(0,0) = 0. \end{cases}$$

Thus, to solve (6.6), it is enough to solve the following equation :

$$(6.10) \qquad h(\xi,\alpha) \equiv \langle G(\lambda_0 + \xi,\ u_0 + \alpha\phi_0 + v),\ \phi_2^*\rangle = 0.$$

For the branch described by (5.15), the tangent direction at (λ_0,u_0) is given by (5.10), namely,

$$(du,d\lambda) = (\phi_2,0).$$

Hence we cannot choose ξ as the parameter to describe this branch. Thus (5.15) can be written as,

$$(6.11) \qquad \begin{cases} \lambda(\alpha) = \lambda_0 + \xi(\alpha), \\ u(\alpha) = u_0 + \alpha\phi_2 + V(\xi(\alpha),\alpha), \end{cases}$$

where,

$$\xi(\alpha) = \frac{1}{2}\frac{d^2\xi}{d\alpha^2}(0)\,\alpha^2 + O(\alpha^3).$$

The coefficient A_0 of (5.15) is given by,

$$A_0 = \frac{1}{2}\frac{d^2\xi}{d\alpha^2}(0) = -\frac{1}{2}\frac{h_{\alpha\alpha\alpha}}{3h_{\alpha\xi}},$$

which is obtained by repeated differentiation of (6.10). For the equation (1.5) , we get,

$$A_0 = \frac{1}{6}\frac{\int f''(u)\ \phi_2^4}{-b} = \frac{1}{6}\frac{\int f''(u)\ \phi_2^4}{2\lambda_0\ \phi'(0)},$$

using (5.11). Except λ_0, the rest are all positive and thus we get,

$$A_0 < 0. \qquad \blacksquare$$

REMARK 6.2 : Following the idea of Pohazaev, we multiply the equation (6.4) for $\hat{u}$, by $x\phi_2'$. The L.H.S. gives,

$$-\int_0^1 (\hat{u})''(x\phi_2') = \int \hat{u}'(x\phi_2')' - [u' \ x \ \phi_2']_0^1$$

$$= \int\{\hat{u}' \ \phi_2' + \hat{u}' \ x \ \phi_2''\} - (\hat{u}'(1).\phi_2'(1))$$

$$= -\int\{\hat{u}'' \ \phi_2 + \hat{u}' \ f'(\hat{u}) \ \phi_2 x\} - \hat{u}'(1).\phi_2'(1)$$

$$= \int_0^1\{(f(\hat{u}) + \hat{\lambda})\phi_2 - (f(\hat{u}))' \ \phi_2 x\} - \hat{u}'(1).\phi_2'(1)$$

$$= \int\{(f(\hat{u}) + \hat{\lambda}) \ \phi_2 + f(\hat{u})(\phi_2' x + \phi_2)\} - \hat{u}'(1).\phi_2'(1).$$

The R.H.S. leads to,

$$\int(f(\hat{u}) + \hat{\lambda})x \ \phi_2' = \int\{f(\hat{u})x \ \phi_2' + \hat{\lambda}x \ \phi_2'\}$$

$$= \int\{f(\hat{u})x \ \phi_2' - \hat{\lambda}\phi_2\}.$$

Now comparing both the sides, we get :

(6.12) $$2 \int(f(\hat{u}) + \hat{\lambda})\phi_2 = \hat{u}'(1).\phi_2'(1).$$

By multiplying (6.3) by $\hat{u}$ and (6.4) by ϕ_2 and subtracting, we have :

(6.13) $$0 = \int(f(\hat{u}) - \hat{u} \ f'(\hat{u}))\phi_2 + \hat{\lambda}\int\phi_2,$$

which yields :

$$\hat{\lambda}\int\phi_2 = (p-1) \int f(\hat{u}) \ \phi_2.$$

Plugging it in (6.12), we obtain,

$$2 \ \{\frac{\hat{\lambda}}{p-1} \ \int\phi_2 + \hat{\lambda} \ \int\phi_2\} = \hat{u}'(1).\phi_2'(1),$$

which results in (6.5). (We recall that $f(u) = |u|^{p-1}u$ and $f'(u) = p|u|^{p-1}$. This is used in (6.13)). In fact, (6.5) is true for any eigenfunction ϕ_n at a singular point (λ_0, u_0) where $\mu_n(-f'(u_0),(0,1)) = 0$. ∎

7 - THE OTHER BIFURCATION POINTS AND THE GLOBAL BEHAVIOUR OF THE BRANCHES S_{2k+1}, $k \geq 1$.

Here we study the case of the second eigenvalue becoming 0, using the same method as was employed in Proposition 5.1 and then the behaviour of the branches S_{2k+1}, $k \geq 1$. The following Corollary is a generalization of Proposition 5.1.

COROLLARY 7.1 : Supposing that at a point (λ_0, u_0) we have $u_0'(0) = u_0'(1) = 0$ and u_0 symmetric, then :

(i) (λ_0, u_0) is a singular point where $\mu_{2n} = 0$ and the corresponding eigen function is given by u_0',

(ii) the bifurcation equation is non-degenerate and has two distinct tangent directions as its roots,

(iii) the two branches meeting at (λ_0, u_0) will be S^-_{2n-1} continuing as S^+_{2n+1} and S^-_{2n} continuing as S^+_{2n} of which the first one only is symmetric with respect to 1/2. (i.e. $S^+_{2n-1} \cup S^-_{2n+1}$ and $S^+_{2n} \cup S^-_{2n}$).

PROOF : Because of the assumptions $u_0'(0) = u_0'(1) = 0$, u_0' will be a non-trivial function in the kernel of $G_u(u_0, \lambda_0)$ and hence (u_0, λ_0) is a singular point. As u_0 is symmetric, u'_0 is antisymmetric with respect to 1/2 and hence it has to be an eigenvalue of even order which is 0 at (u_0, λ_0), say μ_{2n}.

As before, we can derive the bifurcation equation (5.9) where $a = c = 0$ and,

$$b = -\int_0^1 f''(u_0)\phi_{2n}^2 . \Phi = -2\lambda_0 \, \phi'(0),$$

where $\phi_{2n} = u_0'$ is the 2nth eigenfunction and ϕ satisfies the equation (5.12) in which the orthogonality condition is replaced by :

$$\int_0^1 \phi \, \phi_{2n} = 0.$$

By studying the relation (5.13) at a point x where $u(x) = 0$ and $u'(x) = 0$, we see that at that point $\phi(x)$ is also 0, necessarily. At this point the relation,

$$\phi'(x) = - \frac{u_0(x)}{u_0'(x)},$$

by l'Hôspitals rule, gives $\phi'(x) = 0$. Thus x is a double zero for ϕ.

Further at this point, by equation (5.12) :

$$\phi'' = -1 < 0.$$

Thus this point has to be a local maximum.

We also note that if ϕ has a simple zero in (0,1/2), then necessarily $\phi' < 0$ at that point. Thus, either ϕ has only some double zeroes and completely negative or ϕ is positive near the extremities and negative elsewhere with two simple zeroes and a certain number of double zeroes. We will show that the first case is not possible.

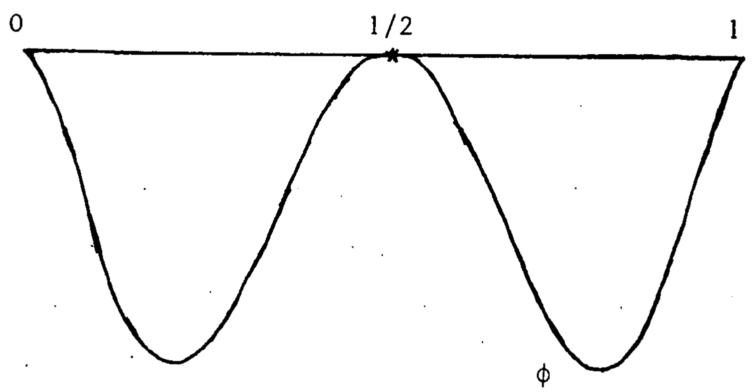


FIGURE 12

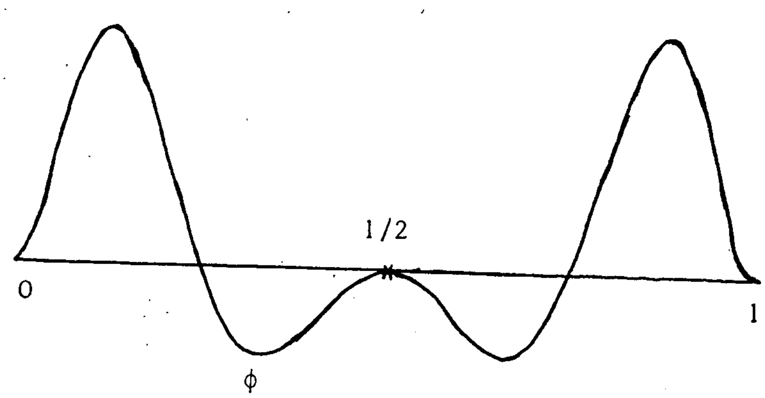


FIGURE 13

For this we need some information on u_0. Indeed u_0 will have to be completely positive if $\lambda_0 < 0$ and negative if $\lambda_0 > 0$, in both the cases except for some double zeroes.

By integrating the equation for u_0, between 0 and x, the first zero of u_0, we see that $u_0'(x) = 0$. Thus every zero of u_0 between 0 and 1 has to be a double zero. To see that u_0 does not change the sign, let us suppose that $\lambda_0 > 0$ and u_0 is also positive in (x_1, x_2) where x_1 and x_2 are two consecutive double zeroes of u_0. Then,

$$\begin{cases} -u_0'' = f(u_0) + \lambda > 0 & \text{in } (x_1, x_2), \\ u_0(x_1) = u_0(x_2) = 0, \end{cases}$$

implies, by maximum principle $\frac{\partial u_0}{\partial n} < 0$ but $u_0'(x_1) = 0$ and $u_0'(x_2) = 0$. This contradiction proves the sign restriction on u_0, depending on λ_0. Thus $(\frac{u_0}{\lambda_0})$ is always a negative function.

Now if ϕ were completely negative, then from (cf. Remark 5.2),

$$2\phi'(0) = \int_0^1 \left(\frac{f''(u_0)}{\lambda_0}\right) \phi_{2n}^2 . \Phi,$$

a contradiction issues because the right hand side is positive whereas $\phi'(0) \leq 0$. Thus the only possibility for ϕ is to be positive near the extremities and negative elsewhere. Then as before by integrating the equation (5.12) between 0 and the first zero y of ϕ' and noting that f' is even,

$$\phi'(0) = \int_0^y f'(u_0)\phi + y > 0,$$

and hence $b \neq 0$ and (ii) is proved.

Proof of part (iii) is as in the earlier case. We can thus get two smooth curves cutting at (λ_0, u_0), given by,

$$\begin{cases} \lambda(t) = \lambda_0 + A_0\, t^2 + O(t^3), \\ u(t) = u_0 + t\, \phi_{2n} + O(t^2); \end{cases}$$

and,

$$\begin{cases} \lambda(t) = \lambda_0 + A_1\, t + O(t^2), \\ u(t) = u_0 + t.\Phi + O(t^2). \end{cases}$$

These expressions for u(t) suggest that the first curve is $S_{2n}^+ \cup S_{2n}^-$ and the second curve is $S_{2n-1}^- \cup S_{2n+1}^+$ or $S_{2n-1}^+ \cup S_{2n+1}^-$. This completes the proof. ∎

REMARK 7.2 : To show these are the only bifurcation points, we have to use an argument similar to the one used for S_2. The crucial point is that we can fabricate a $\hat{u} \in S_{2n+1}$ by extending $u \in S_1$ up to its (2n+1)th zero.

These bifurcation points for $\lambda_0 > 0$ correspond to P_{2n} and for $\lambda_0 < 0$, correspond to Q_{2n}. Thus Theorem 1.1 is proved. We encounter the same difficulty, namely, the determination of exact number of turning points, as in the case of S_2. In any case, we can conclude that there exists at least one tunring point $R^{\pm}_{2n+1}$, for the branch $S^{\pm}_{2n+1}$.

8 - CONCLUSIONS

By restricting ourselves to the case $f = |u|^{p-1}u$ and $g \equiv 1$, we could get a global picture. Instead of $|u|^{p-1}u$, if we let f be any odd function, it is not clear how the other branches S_k will behave although the spiralling is likely to be present. If we drop the oddness assumption on f, then the picture could change drastically. For $f(u) = u^2$, the bifurcation diagram obtained by Ammar KHODJA [1] does not exhibit any spiralling.

Instead of a constant g, if we take a positive function g(x), it seems natural to expect once again the spiralling. In fact the numerical tests performed for the equation,

$$(8.1) \qquad \begin{cases} -u'' = u^3 + \lambda\sin(\pi x) & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

indicate a somewhat similar behaviour, as described in Theorem 1.1, particularly the spiralling and the bifurcation structure.

But here the branch S_3 of our earlier notations, continues as a branch of positive solutions, but now with two humps. Thus there seems to be no hope of proving Proposition 3.2 in this case also, i.e. there may not exist a negative constant $-M$ such that there exist no positive solutions to (8.1) for $\lambda \leq -M$. We conjecture that : *there exist negative constants $\{\lambda_n\}$ such that there exists no positive solution to (8.1), having exactly n humps, for $\lambda < \lambda_n$ and this sequence* $\lambda_n \to -\infty$.

If g changes sign, nothing much is known and the problem is essentially open.

In higher dimension, at least for constant g, we conjecture the existence of a negative constant λ^*, beyond which there are no positive solutions to (1.2). But for other types of g, the problem is open.

PART - II

NUMERICAL STUDY

I. INTRODUCTION

Solving the differential equation numerically amounts to solving the operator equation :

$$(1.1) \qquad G\,(u,\lambda) = 0,$$

where $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. For a nonlinear operator G , we could use Newton's method for example , to obtain the solution of (1.1). Our aim here , is to find smooth branches of solutions (u,λ) of (1.1). It is natural to choose λ as the parameter defining the solution arc $u(\lambda)$ and we can use the continuation procedures , based on Newton's algorithm. Supposing that we know the solution $(\hat{u},\hat{\lambda})$ and the derivative $\frac{du}{d\lambda}(\hat{\lambda})$, then we can set up our iteration scheme as follows :

$$(1.2)\begin{cases} u^{\circ}\,(\lambda+\delta\lambda) = \hat{u} + \delta\lambda\,\dfrac{du}{d\lambda}\,(\lambda)\,, \\ G_u\,(u^n(\hat{\lambda}+\delta\lambda),\ \hat{\lambda}+\delta\lambda)\ \delta u^n(\hat{\lambda}+\delta\lambda) = -\,G\,(\,u^n(\hat{\lambda}+\lambda\delta),\ \hat{\lambda}+\delta\lambda), \\ u^{n+1}(\hat{\lambda}+\delta\lambda) = u^n(\hat{\lambda}+\delta\lambda) + \delta u^n(\hat{\lambda}+\delta\lambda)\,. \end{cases}$$

But this scheme runs into difficulty at a singular point , where G_u is no more invertible. To overcome this problem , we can choose a new parameter s different from λ and add a normalisation condition :

(1.3) $$N\,(u,\lambda,s) = 0,$$

where $N : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}$. Let $U \in \mathbb{R}^{n+1}$ and H be the operator from $\mathbb{R}^{n+1} \times \mathbb{R}$ into $\mathbb{R}^{n+1}$ defined by.

(1.4) $$U \equiv (u,\lambda) \quad ; \; H\,(U,s) \equiv \begin{pmatrix} G\,(u,\lambda) \\ N\,(u,\lambda,s) \end{pmatrix}.$$

Now we search for the solution :

$$U\,(s) = (u(s),\lambda(s)),$$

satisfying,

(1.5) $$H\,(U(s),s) = 0.$$

For a fixed s, the solution x(s) is isolated if

$$\mathcal{A}\,(s) \equiv H_U\,(U(s),s) = \begin{pmatrix} G_u(u(s),\lambda(s)) & G_\lambda(u(s),\lambda(s)) \\ N_u(u(s),\lambda(s),s) & N_\lambda(u(s),\lambda(s),s) \end{pmatrix}$$

is non-singular. Even if G_u is singular, with a convenient choice of N, the inflated matrix can be made non-singular. The following lemma (cf.[11] B) indicates such possibilities :

<u>LEMMA 1.1</u> : Let $\mathbb{B}$ be a Banach space and consider the linear operator $\mathcal{A} : \mathbb{B} \times \mathbb{R}^n \to \mathbb{B} \times \mathbb{R}^n$ of the form :

$$\mathcal{A} \equiv \begin{pmatrix} A & B \\ C^* & D \end{pmatrix} \text{where} \quad \begin{cases} A : \mathbb{B} \to \mathbb{B}, \; B : \mathbb{R}^n \to \mathbb{B}\,; \\ C^* : \mathbb{B} \to \mathbb{R}^n, \; D : \mathbb{R}^n \to \mathbb{R}^n. \end{cases}$$

(i) If A is non-singular then $\mathcal{A}$ is non-singular iff :

$$D - C^* A^{-1} B \text{ is non-singular.}$$

(ii) If A is singular and

$$\text{Dim}\,(\ker(A)) = \text{codim}\,R\,(A) = n$$

then $\mathcal{A}$ is non-singular iff :

(C_0) Dim R (B) = n $\quad (C_1)$ $R\,(B) \cap R\,(A) = 0$

(C_2) Dim R (C^*) = n $\quad (C_3)$ $\ker\,(A) \cap \ker\,(C^*) = 0$.

(iii) If A is singular and Dim (ker(A)) > n, then $\mathcal{A}$ is singular. ∎

Hence even if G_u is singular, once the inflated matrix is made non-singular, then with a Newton's scheme adapted to the equation (1.5), we can pass over certain singular points like a turning point or a simple bifurcation point. Following the idea proposed in GLOWINSKI, KELLER & REINHART [10], we use the above continuation procedure, in combination with conjugate gradient techniques after recasting the equation (1.5) in a suitable least square formulation. This often results in a more robust numerical scheme.

2. DESCRIPTION OF THE NUMERICAL SCHEME

Let the functional J be defined by

$$(2.1) \qquad J(u,\lambda) \equiv \frac{1}{2} (A\tilde{u},\tilde{u}) + \frac{1}{2} a \tilde{\lambda}^2$$

where

(i) the matrix A is chosen conveniently so as to be symmetric positive definite

(ii) $\tilde{u}$ is given by,

$$(A\tilde{u},w) = (G(u,\lambda),w) \quad \forall\, w \in \mathbb{R}^n,$$

(iii) a is a positive constant, chosen appropriately,

(iv) $\tilde{\lambda}$ is defined using the normalisation function N,

$$a\tilde{\lambda} = N(u,\lambda).$$

For example, we could choose

$$\begin{cases} A = \text{discretized matrix corresponding to } -\Delta, \\ N(u,\lambda) = \|\dot{u}\|^2 + |\dot{\lambda}|^2 - 1. \end{cases}$$

A least square formulation of (1.5) is :

$$(2.2) \quad \begin{cases} \text{Find } U(s) = \{u(s),\lambda(s)\} \text{ such that} \\ J(U(s)) \leq J(W) \quad \forall\, W = \{w,\mu\} \in \mathbb{R}^{n+1} \end{cases}.$$

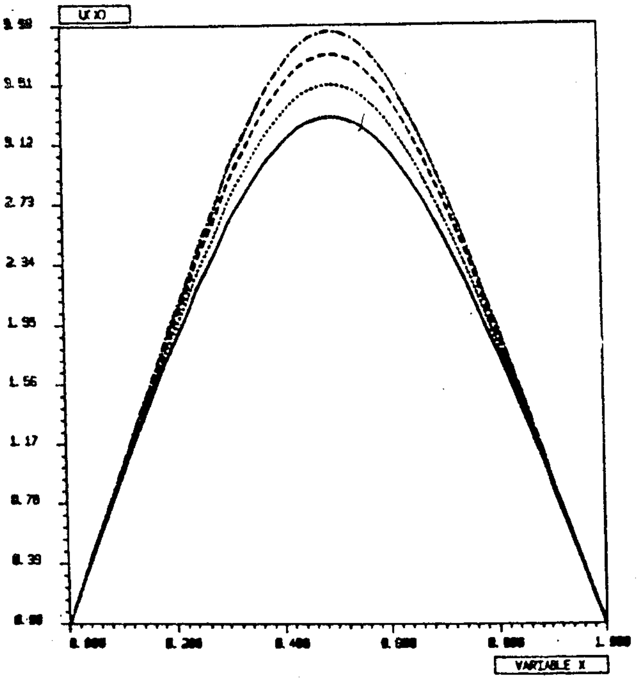
Using this formulation of the problem, we can use the standard conjugate gradient algorithm to minimize $J(u,\lambda)$, after having calculated the partial derivatives, J_u and J_λ. (Refer to [10]).

The continuation procedure can be started either with one solution $(\hat{u},\hat{\lambda})$ and the derivative at that point $\frac{du}{d\lambda}(\hat{\lambda})$ or with 2 neighbouring solutions (u_o,λ_o) and (u_1,λ_1). For more details about the implementation of this algorithm, one can refer, for example [18] and [10].

Symmetric solutions of the equation :

$$\begin{cases} -u'' = u^3 + \lambda \text{ in } (0,1), \\ u(o) = u(1) = o \end{cases}$$

obtained by the continuation procedure starting from the zero solution for $\lambda = o$:



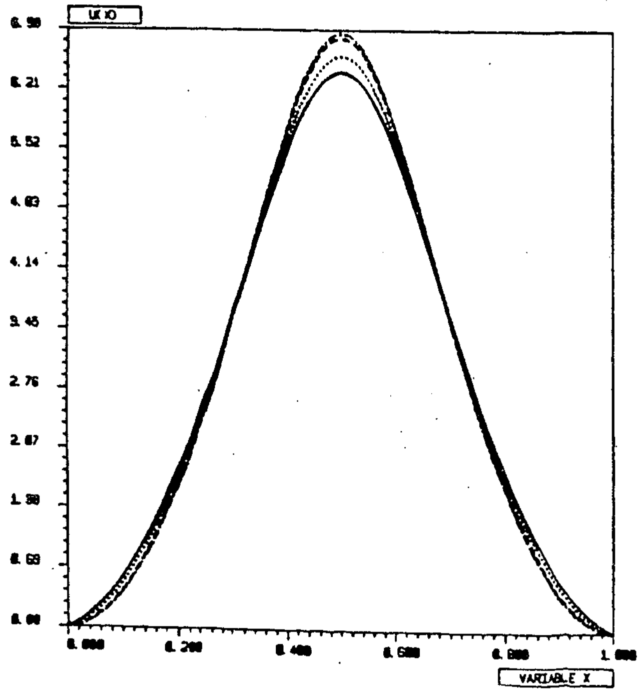
(1)

———— $\lambda = 0.5$

. . . . $\lambda = 0.3$

- - - - - $\lambda = 0.0$

- . - . - . $\lambda = -0.2$



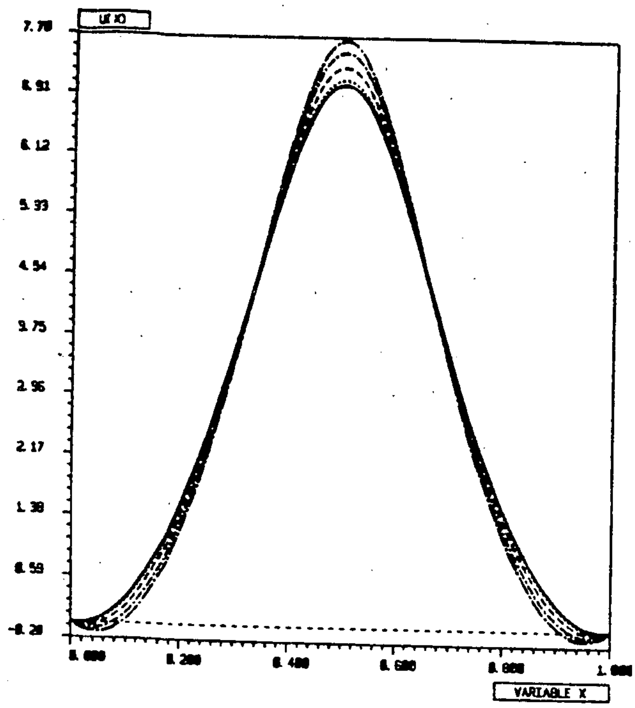
(2)

———— $\lambda = -6.5$

. . . . . . $\lambda = -7.2$

- - - - - $\lambda = -7.9$

- . - . - . $\lambda = -8.1$



(3)

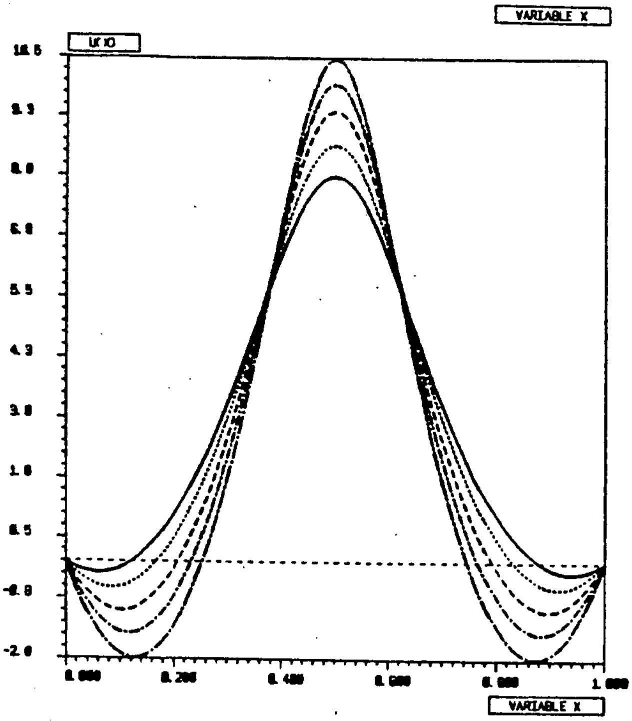
λ = - 8.8

λ = - 9.1

λ = - 9.7

λ = -10.4

λ = -11.1



(4)

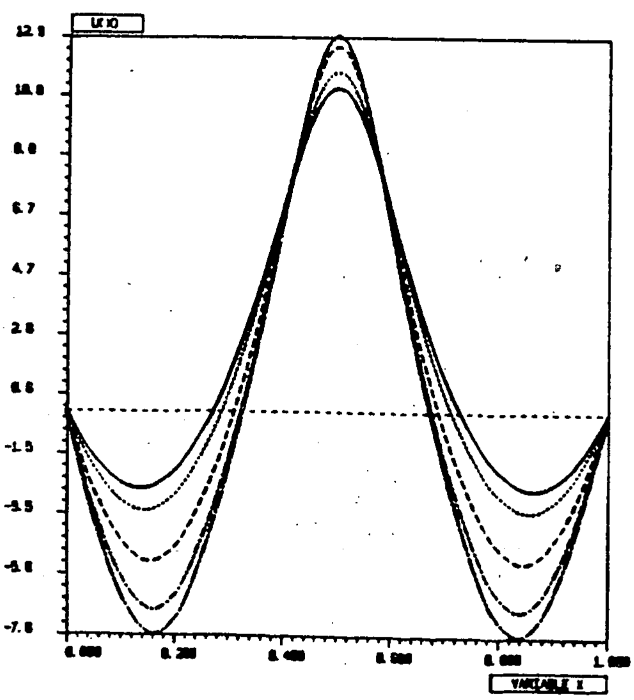
λ = - 12.4

λ = - 15.2

λ = - 18.6

λ = - 21.4

λ = - 24.2



(5)

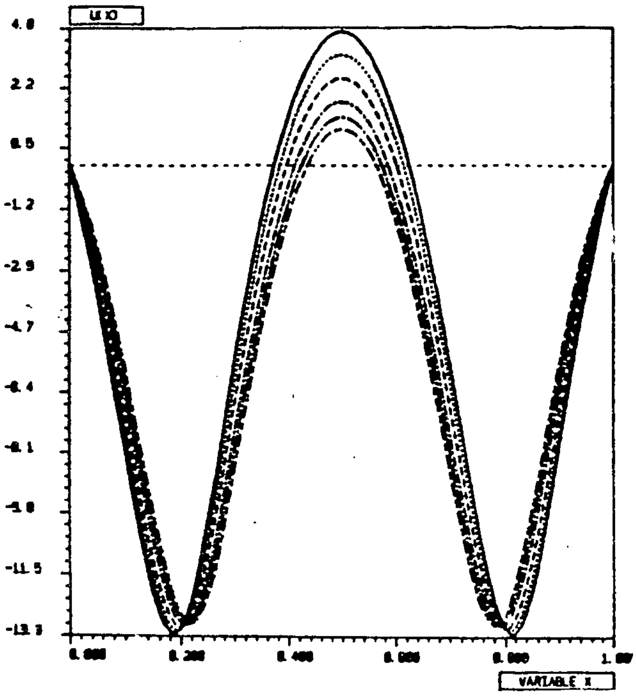
λ = - 27.3

λ = - 30.2

λ = - 33.5

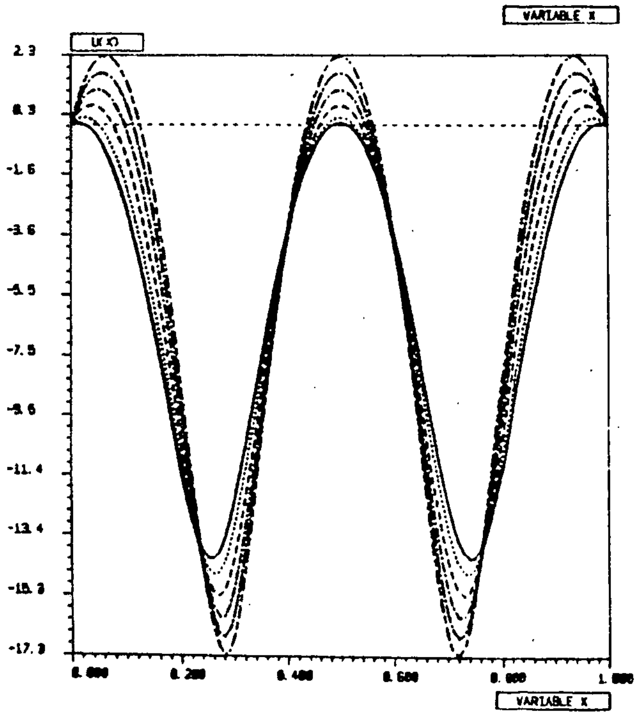
λ = - 32

λ = - 29.3



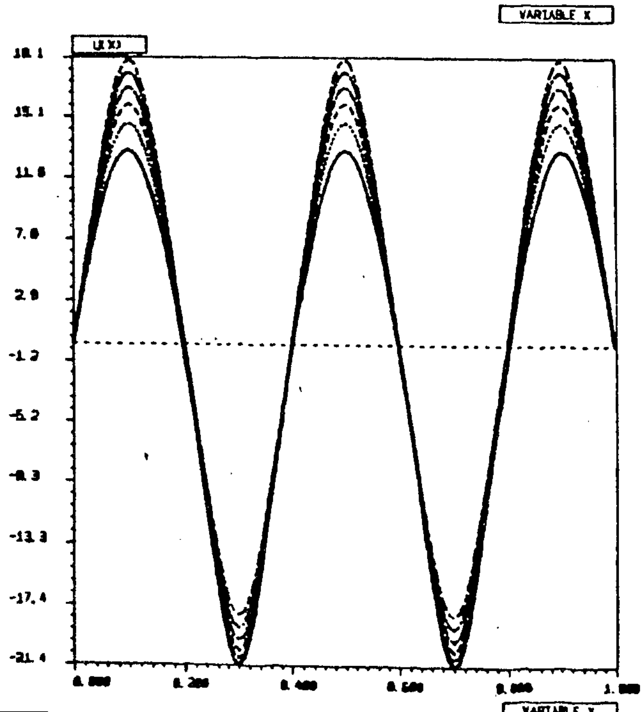
(6)

$\lambda = 44.3$
$\lambda = 45.3$
$\lambda = 46.2$
$\lambda = 47.5$
$\lambda = 48.6$
$\lambda = 49.9$



(7)

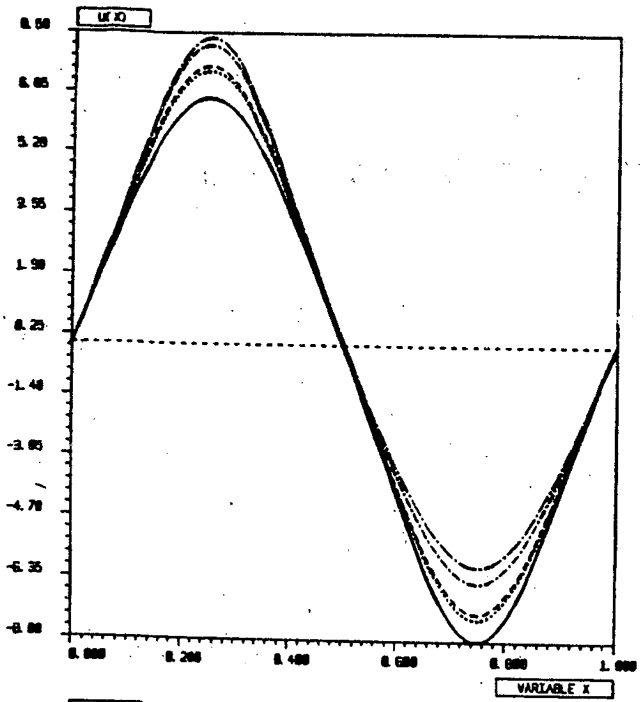
$\lambda = 70.5$
$\lambda = 77.4$
$\lambda = 87.0$
$\lambda = 96.9$
$\lambda = 105.9$
$\lambda = 114.1$



(8)

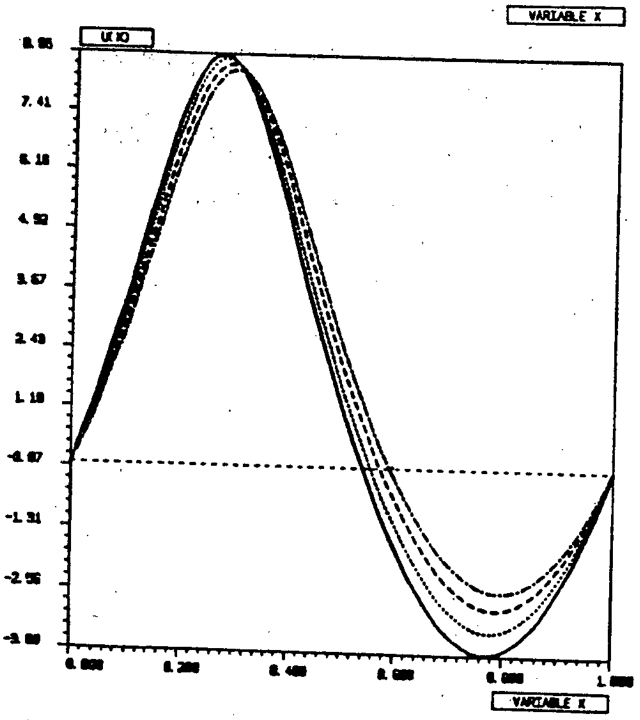
$\lambda = 135.3$
$\lambda = 105$
$\lambda = 76.1$
$\lambda = 45.2$
$\lambda = 12.3$
$\lambda = -16.8$

Solutions with one node (in particular the branch S_2^+)



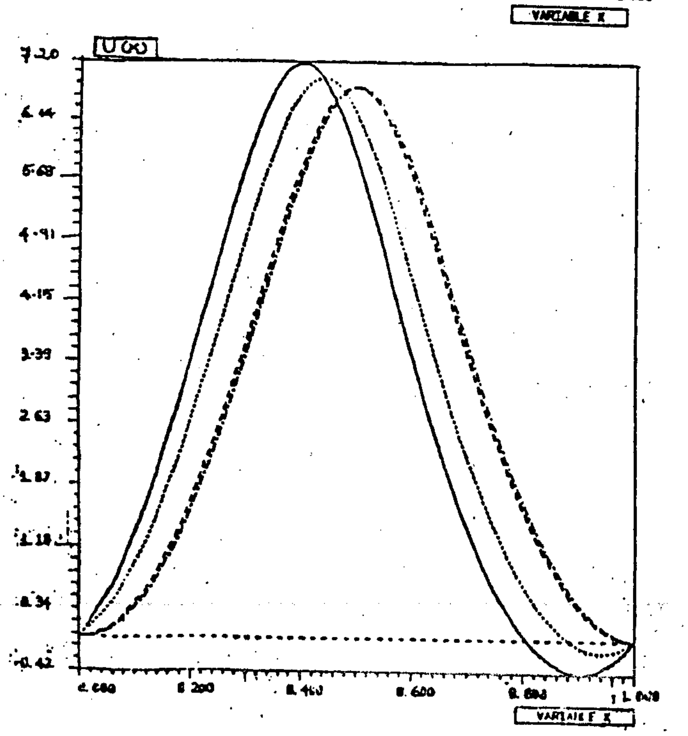
(9)

———————	$\lambda = 3.7$
• • • • • • •	$\lambda = 0.0$
— — — —	$\lambda = -.001$
— • — • — •	$\lambda = -4.6$
—— • —— •	$\lambda = -6.2$



(10)

———————	$\lambda = -10.8$
• • • • • •	$\lambda = -11.01$
— — — —	$\lambda = -11.04$
—— • — • —	$\lambda = -10.9$

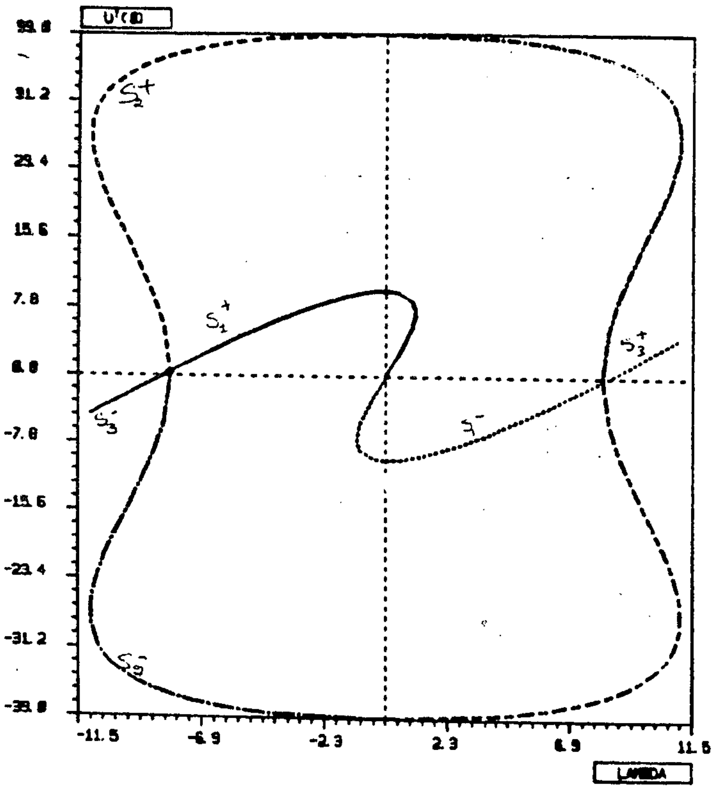


(11)

———————	$\lambda = -8.7$
• • • • • • •	$\lambda = -8.3$
— — — —	$\lambda = -8.11$
— • — • — •	$\lambda = -8.12$

The bifurcation diagram (namely, λ versus $u'(o)$) for the equation :

$$\begin{cases} - u'' = u^3 + \lambda \quad \text{in } (0,1), \\ u(o) = u(1) = o. \end{cases}$$



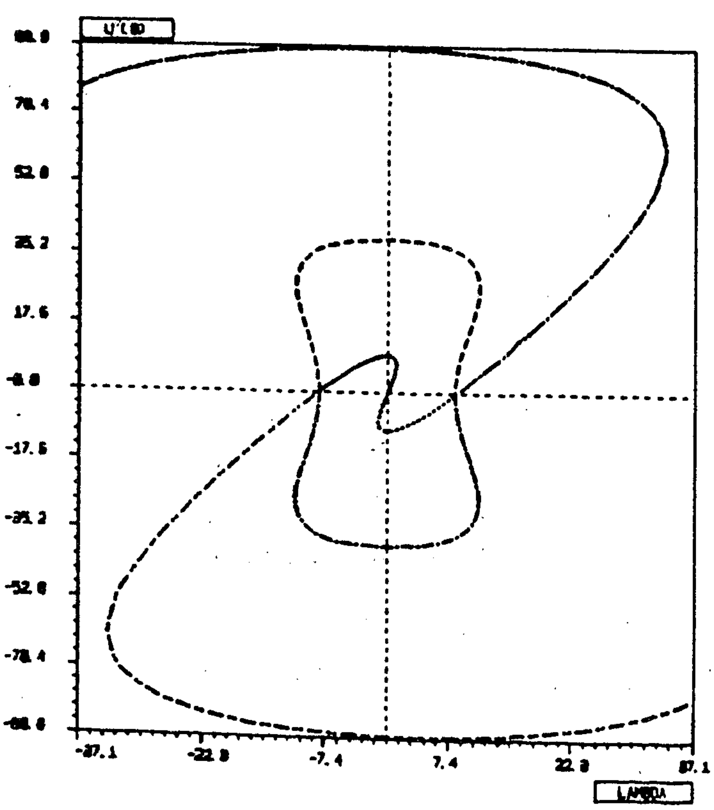
(1)

positive solutions S_1^+

négative solutions S_1^-

S_2^+ } Solutions with

S_2^- } one node.



(2)

S_1^+

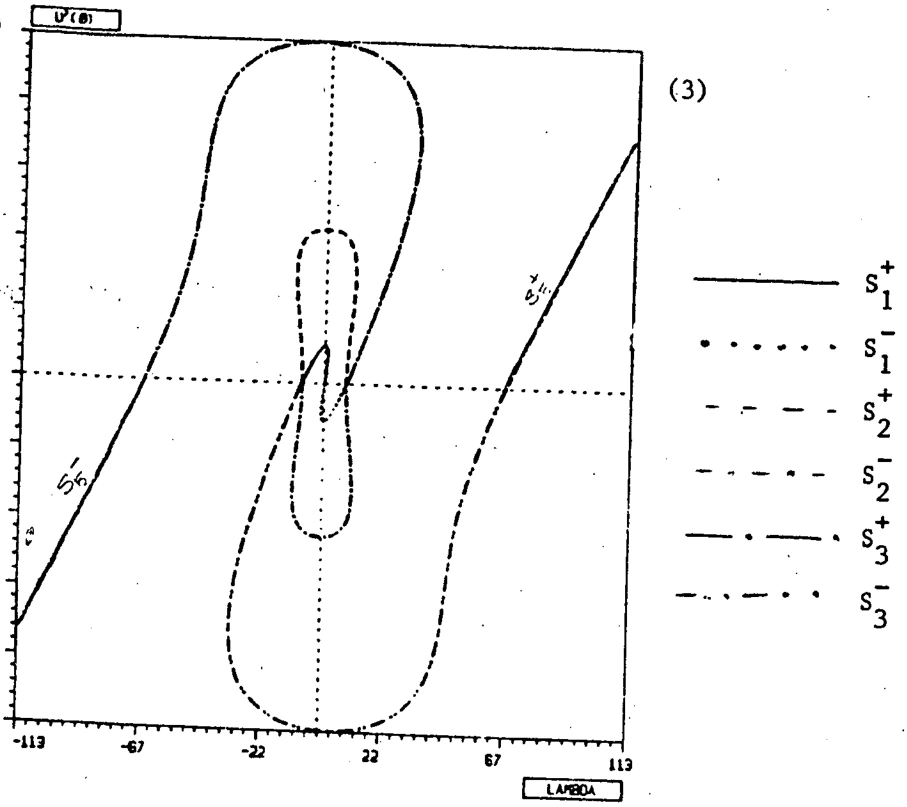
S_1^-

S_2^+

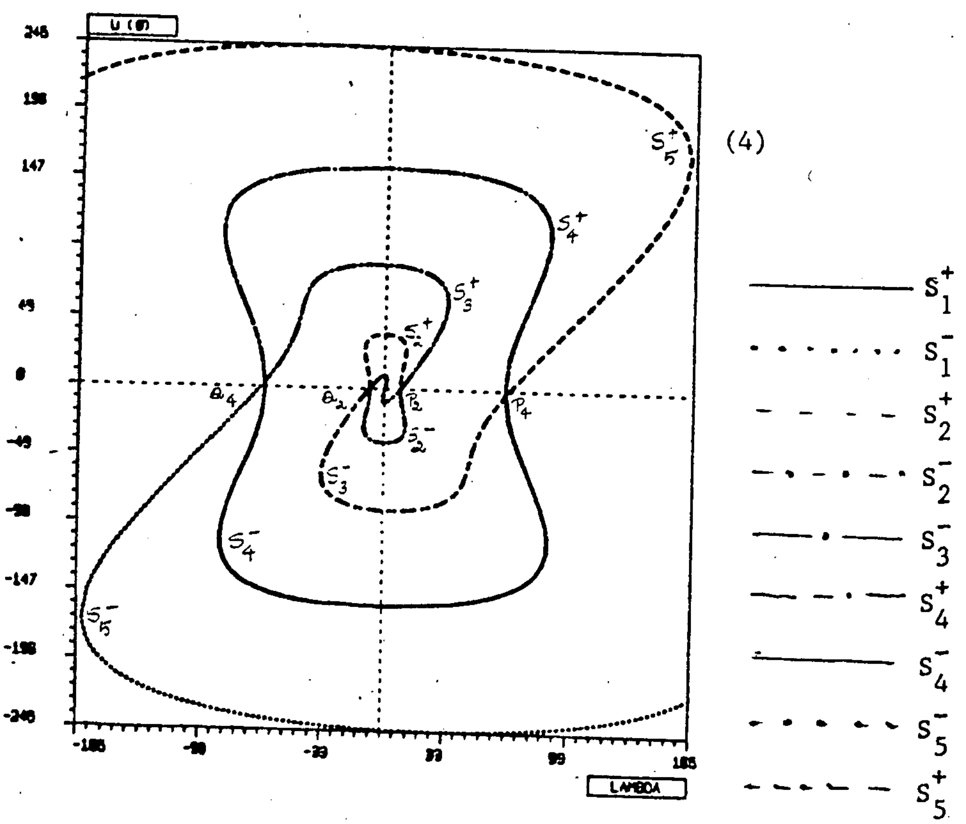
S_2^-

S_3^+

S_3^-


U'(θ)
88.0
70.4
52.8
35.2
17.6
-0.0
-17.6
-35.2
-52.8
-70.4
-88.0
-113
-67
-22
22
67
113
LAMBDA
S_5^-
S_5^+
S_1^+
S_1^-
S_2^+
S_2^-
S_3^+
S_3^-

(3)

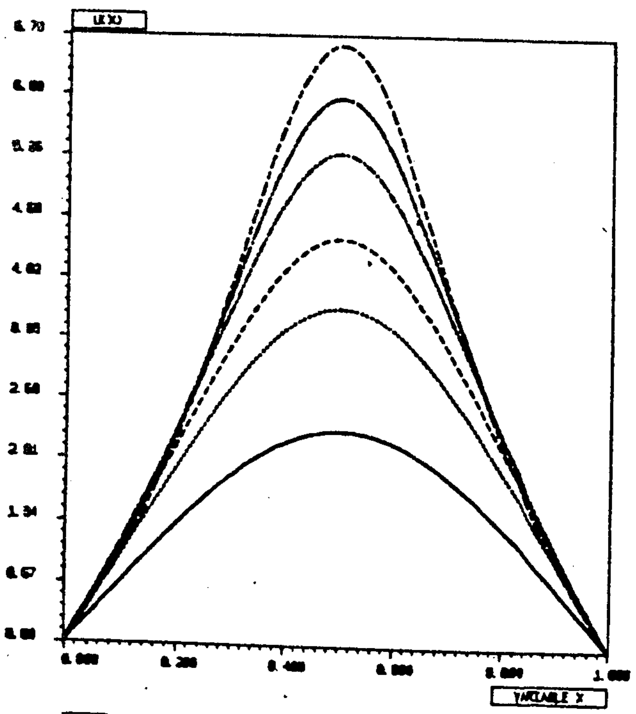

U(θ)
246
196
147
49
0
-49
-147
-196
-246
-165
-33
33
99
165
LAMBDA
S_5^+
S_4^+
S_3^+
S_2^+
S_2^-
S_3^-
S_4^-
S_5^-
a_4
a_2
P_2
P_4
S_1^+
S_1^-
S_2^+
S_2^-
S_3^-
S_4^+
S_4^-
S_5^-
S_5^+

(4)

Symmetric solutions of the equation

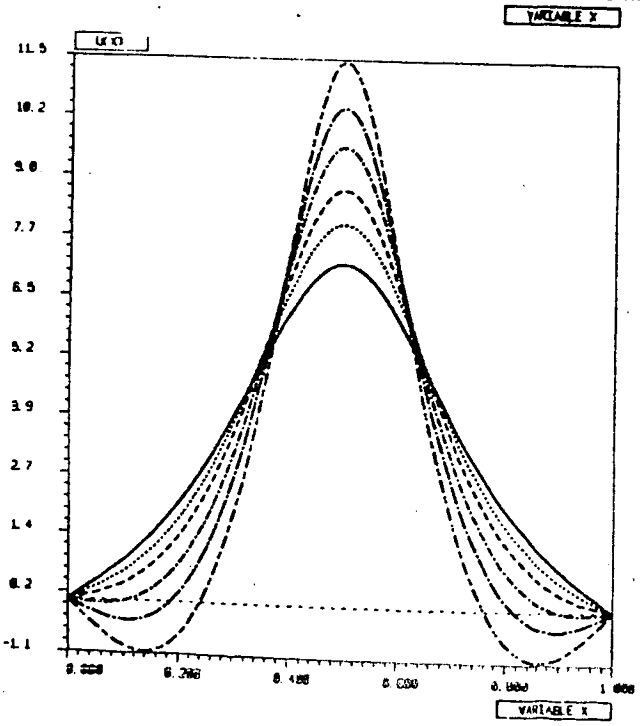
$$\begin{cases} -u'' = u^3 + \lambda \sin(\pi x) \text{ in } (0,1) \\ u(o) = u(1) = o. \end{cases}$$

obtained by the continuation procedure starding from the zero solution for $\lambda = o$.



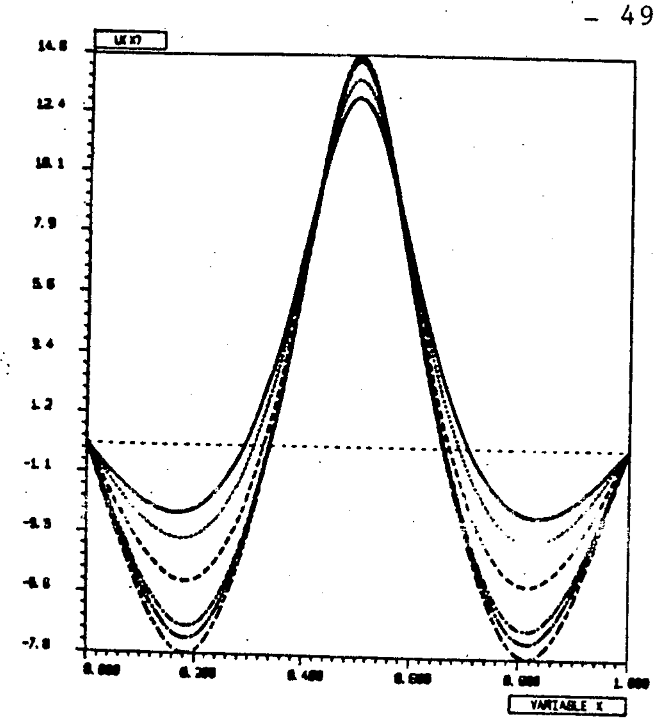
(1)

————————	$\lambda = 1.3$
- - - - - - - -	$\lambda = 0.03$
— — — — —	$\lambda = -1.7$
— - — - — - —	$\lambda = -4.7$
—— · —— ·	$\lambda = -7.2$
— · · — · · —	$\lambda = -9.8$



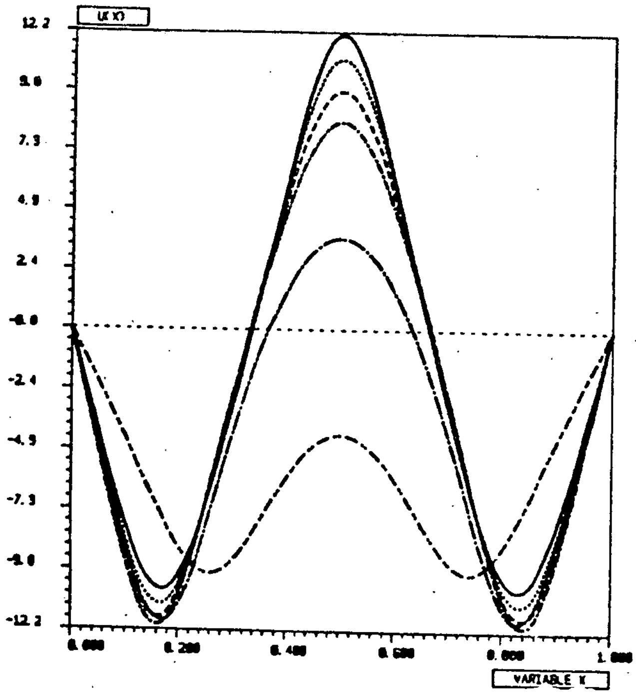
(2)

————————	$\lambda = -12.9$
· · · · · · · ·	$\lambda = -17.2$
— — — — —	$\lambda = -21.2$
— · — · — · —	$\lambda = -26.7$
—— · —— ·	$\lambda = -32.2$
— · · — · ·	$\lambda = -39.3$



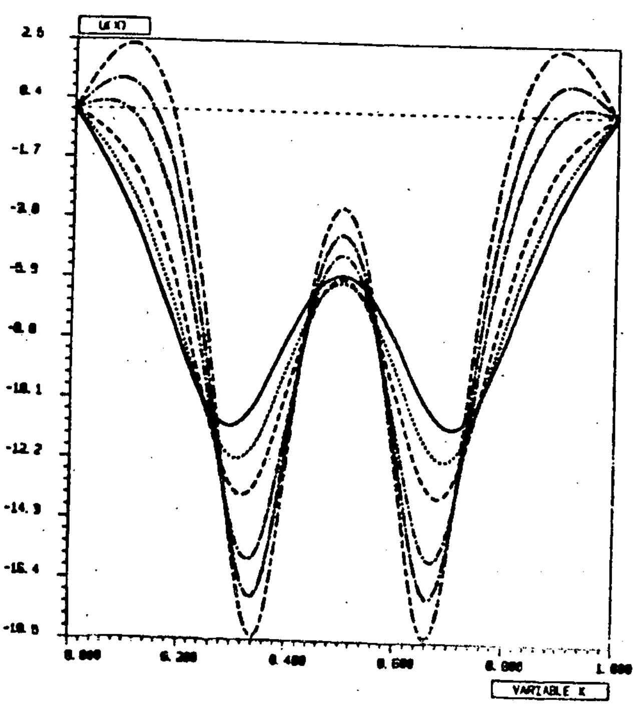
(3)

λ = - 51.4
λ = - 56.6
λ = - 60.5
λ = - 56.6
λ = - 53.9
λ = - 49.8



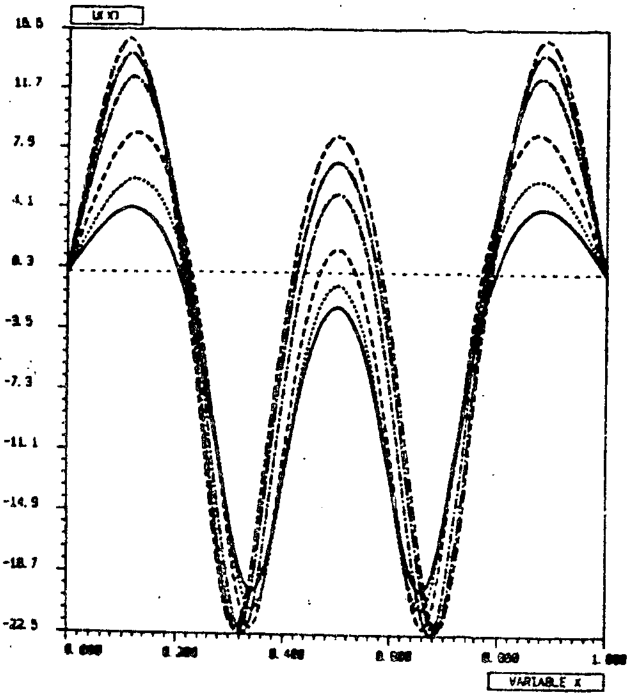
(4)

λ = - 11.6
λ = 0.5
λ = 13.7
λ = 23.4
λ = 38.7
λ = 45.3



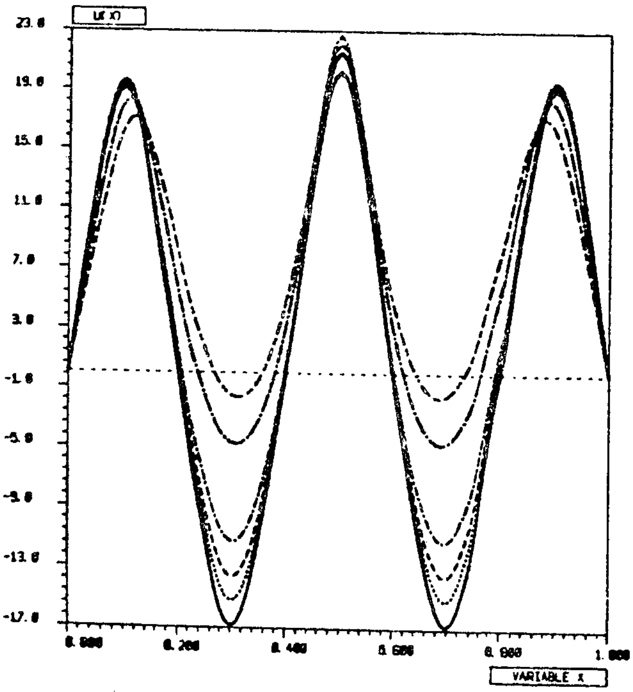
(5)

λ = 70.0
λ = 89.6
λ = 110.9
λ = 148.6
λ = 172.2
λ = 199.9



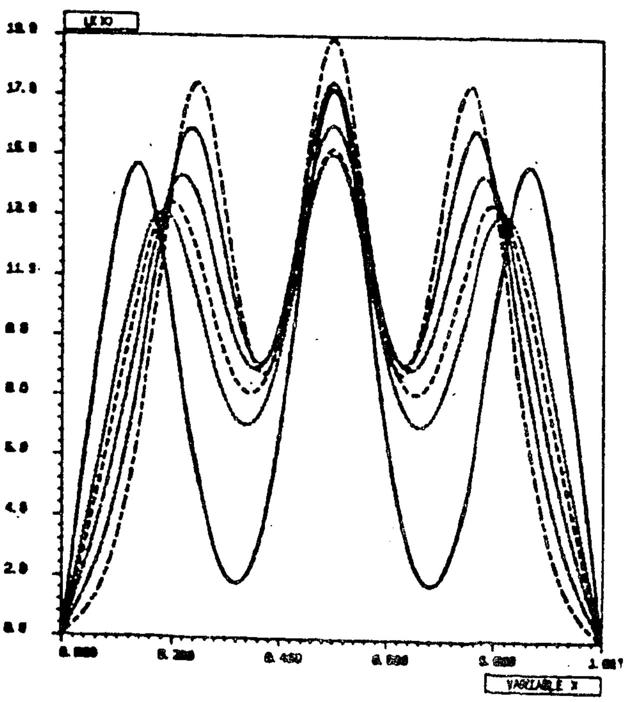
(6)

————————	λ = 229.5
···········	λ = 252.2
– – – – –	λ = 266.6
–·–·–·–	λ = 246.3
—— · ——	λ = 226.3
–··–··	λ = 207.7



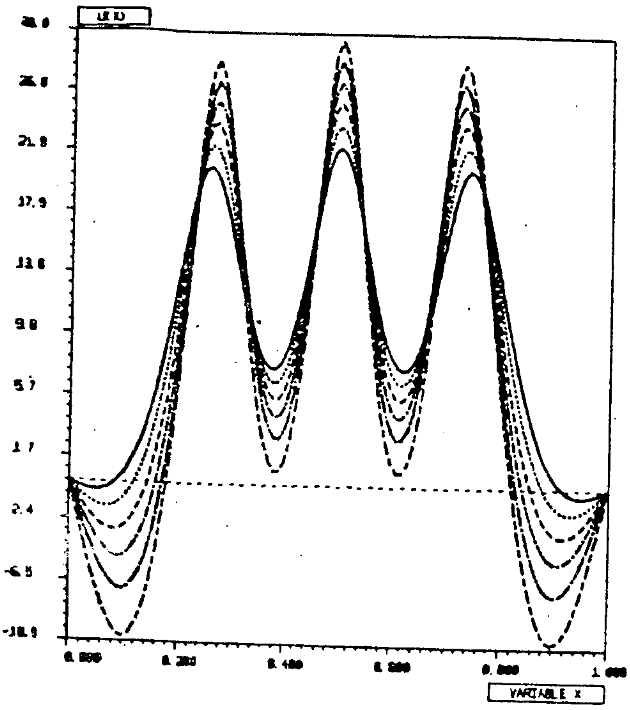
(7)

————————	λ = - 64.1
···········	λ = - 114.4
– – – – –	λ = - 152.6
–·–·–·–	λ = - 192.9
—— · ——	λ = - 214.6
–··–··	λ = - 195.6



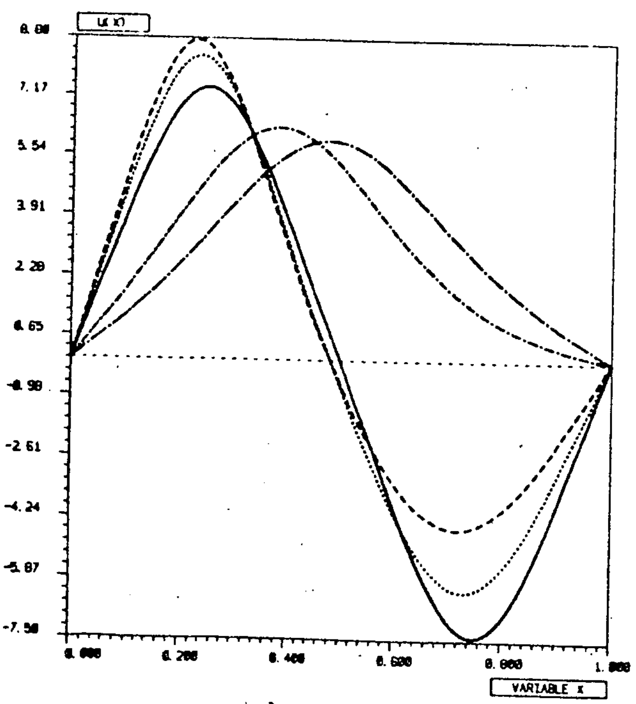
(8)

————————	λ = - 172.8
···········	λ = - 182.2
– – – – –	λ = - 291.8
–·–·–·–	λ = - 243.9
—— · ——	λ = - 296.3
–··–··	λ = - 344.9

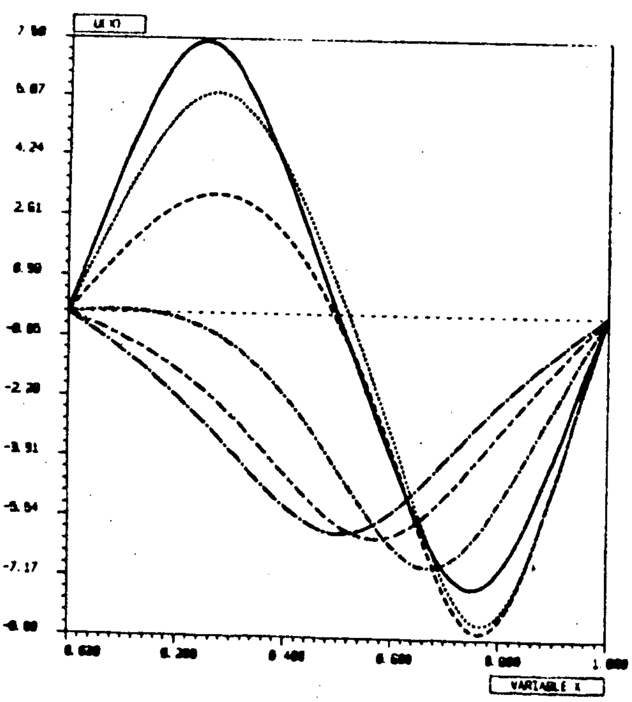


λ = - 424.5
λ = - 475.4
λ = - 531.2
λ = - 582.9
λ = - 634.6
λ = - 680.4

Solution with one node (in particular, the branch S_2^+) :



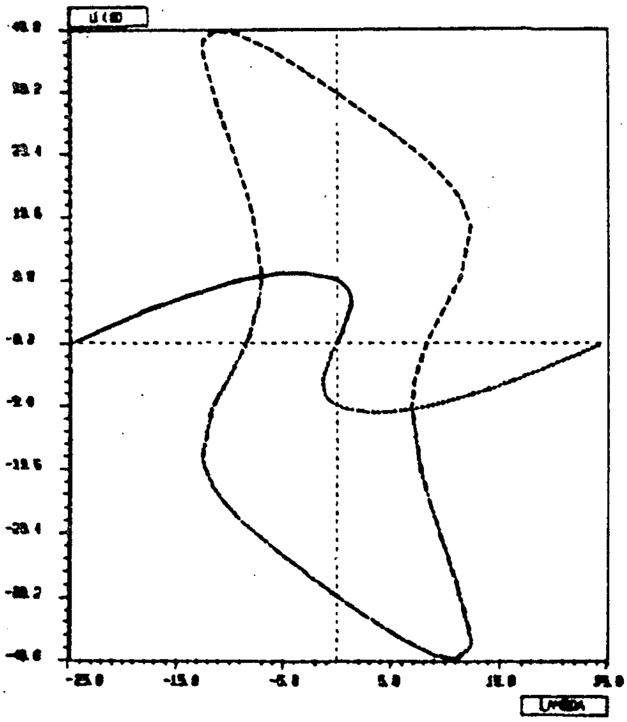
λ = -0.01
λ = -6.1
λ = -11.2
λ = -7.6
λ = -7.0



λ = 0.0
λ = 7.0
λ = 12.5
λ = 8.6
λ = 7.0
λ = 7.2

The bifurcation diagram (namely λ versus $u'(0)$) for the equation :

$$\begin{cases} -u'' = u^3 + \lambda \sin(\pi x) & \text{in } (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

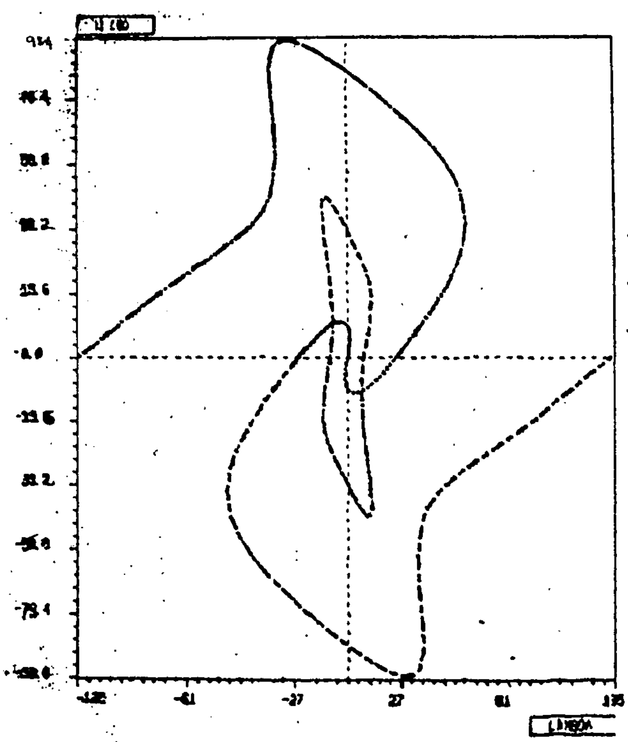


——— positive solutions : S_1^+

······· negative solutions : S_1^-

– – – – - solutions with one node : S_2^+

–·–·– solutions with one node : S_2^-



——— S_1^+ : solutions with one maximum

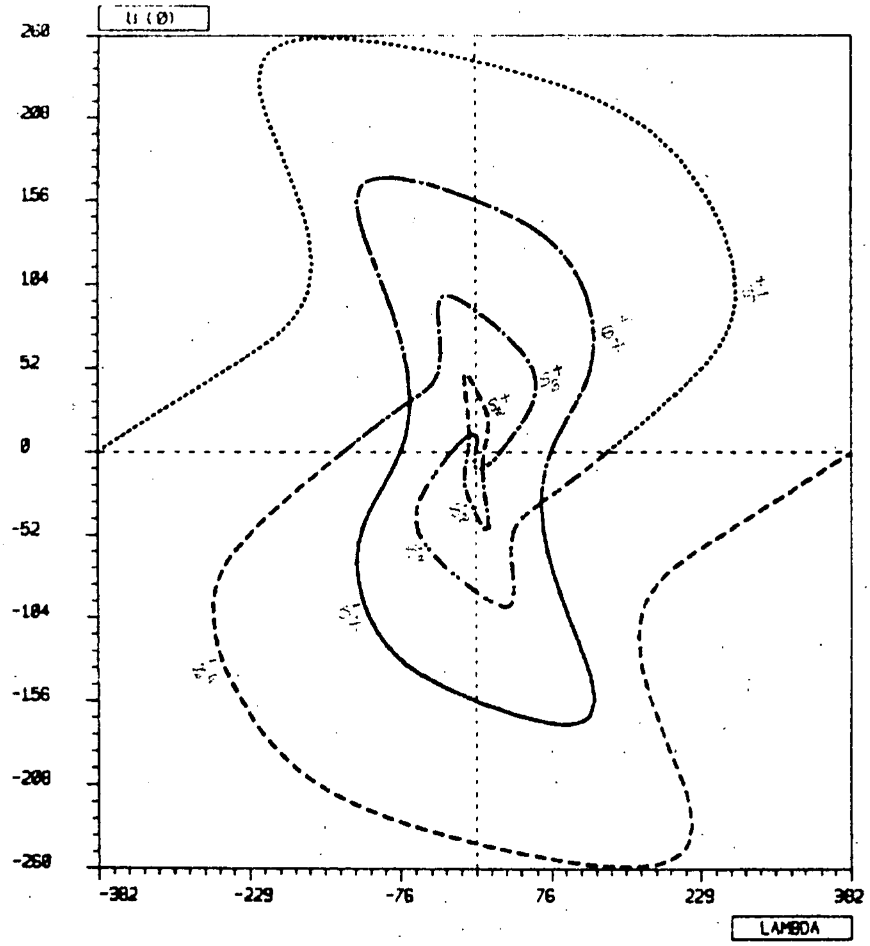
····· S_1^- : solutions with one minimum

– – – S_2^+ : solutions with one maximum and one minimum

·–· S_2^- : solutions with one minimum and one maximum

——— S_3^+ : solutions with 2 maxima and one minima

–·· S_3^- : solutions with 2 minima and one maxima


U (0)
260
208
156
104
52
0
-52
-104
-156
-208
-260
-382
-229
-76
76
229
382
LAMBDA

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Imprimé en France

par

l'Institut National de Recherche en Informatique et en Automatique